

Boundary fermions and the plane wave

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Abstract

We construct branes in the plane wave background under the inclusion of fermionic boundary fields. The resulting deformed boundary conditions in the bosonic and fermionic sectors give rise to new integrable and supersymmetric branes of type (n, n) . The extremal case of the spacetime filling $(4, 4)$ -brane is shown to be maximally spacetime supersymmetric.

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1 Introduction

In recent years, fermionic boundary fields have been used in a number of different field theoretical settings. Soon after the introduction of boundary fermions for the massive Ising model defined on a manifold with boundaries in [1] in the context of integrable boundary models, Warner initiated in [2] their application in the study of $\mathcal{N} = (2, 2)$ supersymmetric Landau-Ginzburg models. In this case, terms corresponding to the boundary fermions are chosen to cancel a generically nonzero boundary contribution (the so called Warner term) in the supersymmetry variation of the bulk Lagrangian in the presence of D-branes. In the context of integrable boundary field theories similar ideas have thereafter been used in [3, 4] to construct boundary conditions compatible with supersymmetry and integrability for extensions of the sine-Gordon model. In the string community they finally led to the study of matrix factorisations and the relation of D-brane physics and coherent sheaves, compare for example with [5, 6, 7] as initial references.

Although we will use a Lagrangian analogously to those appearing in [4, 7, 6], we want to mention that similar boundary fermionic fields also appeared in constructions aiming at nonabelian extensions of Dirac-Born-Infeld D-brane descriptions. Going back to [8], the boundary fermions are in this case interpreted as representing Chan-Paton factors. A discussion along these lines in the context of pure spinors is presented in [9] and a treatment using the Green-Schwarz formulation is to be found [10].

The relation of plane wave physics to boundary fermionic fields in the context of $\mathcal{N} = 2$ supersymmetry is most easily derived from the work of Maldacena and Maoz in [11] on nontrivial Ramond-Ramond type II B supergravity backgrounds, chosen to preserve at least 4 spacetime supersymmetries. For a flat transverse space these backgrounds of pp-wave structure are exact superstring solutions [12] and in this case parametrised by a single holomorphic function. In the corresponding worldsheet theory, given by a $\mathcal{N} = (2, 2)$ supersymmetric Landau-Ginzburg model, this function becomes the worldsheet superpotential $W(\mathbf{z})$. For further applications of methods from [12] to comparable backgrounds as constructed in [11], see for example [13].

The choice of a trigonometric superpotential $W(z) \sim \cos z$ in the solutions of [11] leads to the integrable $\mathcal{N} = 2$ supersymmetric sine-Gordon model on the worldsheet, whereas the exponential $W(z) \sim e^z$ gives rise to the $\mathcal{N} = 2$ Liouville theory. In the context of boundary fermions these theories have been discussed in [4, 14, 15] and [16].

Using an approach in this spirit, but see also the Lagrangians defined in [7, 6], we will be interested in this paper in the situation described by the superpotential

$$W(\mathbf{z}) = -im \sum_{j=1}^4 (z^j)^2. \quad (1)$$

As pointed out for example in [11], one reobtains from (1) the situation of strings in the maximally supersymmetric type II B plane wave background from [17], described by the metric

$$ds^2 = 2dX^+dX^- - m^2 X^I X^I dX^+dX^+ + dX^I dX^I \quad (2)$$

and the nontrivial five-form components

$$F_{+1234} = F_{+5678} = 2m. \quad (3)$$

The closed string theory in this background was first solved in [18] and attracted a substantial amount of interest in particular after the appearance of [19] linking the string theory on (2) to the general study of the AdS/CFT correspondence from [20]. For reviews of this field see [21].

Branes in the plane wave background of [17] have been studied in a number of papers from different point of views. We will briefly review them and the classification of the maximally supersymmetric branes into class I and II branes from [22, 23] in section 3 where we will also point to the corresponding literature.

Here we only want to mention that all the maximally supersymmetric branes known for this background are also integrable, that is, they preserve the integrable structure of the closed string theory in the sense of [1]. Handling a free theory, relatively little attention is usually payed to this point. However, the inclusion of boundary fermions modifies this situation as they generically give rise to an interacting boundary field theory which is in most cases also incompatible with integrability. The requirement of conserved higher spin currents in the boundary theory will lead to strong constraints on admissible boundary couplings. It is worth mentioning that the massive Ising model, appearing as the fermionic part of the plane wave worldsheet theory of (2), has been intensively discussed in the literature on integrable (boundary) models, see for example [1, 24] and references therein.

From the point of view of the $\mathcal{N} = 2$ worldsheet supersymmetry, branes in Maldacena-Maoz backgrounds have been studied in [25], building on the work of [26]. In this case the “Warner problem” is avoided by the choice of particular (oblique) orientations of

the Neumann directions, implying a vanishing Warner term. The oblique branes of [25] for the particular plane wave background have been studied along the lines of [27, 22, 23] in [28].

In this paper we discuss branes beyond the restrictive setting of [25] by aiming first of all at integrable branes with preserved $\mathcal{N} = 2$ (worldsheet) supersymmetry under the inclusion of fermionic boundary fields. In the classification of [22, 23] the new branes are all of type (n, n) and as a main result, the limiting case of the spacetime filling $(4, 4)$ -brane with only Neumann directions in the transverse space is found to be maximally spacetime supersymmetric. This is in analogy to the other limiting case of the $(0, 0)$ instanton from [22, 23], with which it also shares analogous boundary state overlaps.

The bosonic boundary conditions of the new branes are expressible as a standard coupling to a nonzero longitudinal flux \mathcal{F}_{+I} . The fermionic bulk and boundary fields, on the other hand, are first of all determined due to a coupled system of differential equations on the boundary. The on-shell elimination of the boundary fermions from this system leads to an expression for the bulk field boundary conditions in terms of a linear differential equation in the boundary parametrising coordinate τ . As an interesting result, the boundary fermions can finally be expressed as a function of the bulk fermionic fields without including additional degrees of freedom. In the quantum theory the corresponding expressions also correctly reproduce the required quantum mechanical anticommutation relations for the boundary fields.

This paper is organised as follows. In the starting section 2 we collect background information on the plane wave theory formulated as a Landau-Ginzburg model and state the relation between the $\mathcal{N} = (2, 2)$ worldsheet supercharges and the maximal spacetime supersymmetry from [17, 18]. After briefly reviewing branes in the plane wave theory in section 3, we start in section 4 our study of boundary fermions in the context of plane wave physics and derive the conditions for integrable and $\mathcal{N} = 2$ supersymmetric branes. The branes solving these conditions are then studied in detail in section 5 by constructing and quantising the corresponding open string theory. In the subsequent section 6 we conduct a discussion using boundary states, leading in particular to a study of preserved spacetime supersymmetries in the presence of boundary fermions. Here we also briefly suggest, following [29], how to realise the deformed Neumann boundary conditions in the bosonic sector by nonzero longitudinal fluxes. In the final section 7, the equivalence of the open and closed string constructions is discussed along the lines of [30, 23] by establishing the equality of certain open string partition functions with the corresponding closed string boundary state overlaps. Certain technical details are collected in the appendices.

2 The plane wave as a Landau-Ginzburg model

In this section we collect some information about the worldsheet theory for strings in the maximally supersymmetric plane wave background of [17] formulated as a $\mathcal{N} = (2, 2)$ supersymmetric Landau-Ginzburg model. In particular, we mention the relation between the Landau-Ginzburg and Green-Schwarz fermions along the lines of [11]. This will especially also lead to expressions for the $\mathcal{N} = (2, 2)$ supercharges as linear combinations of the spacetime supersymmetries from [17, 18]. As these results are crucial for the later sections, we supply some additional details in the appendix A. Our conventions for Landau-Ginzburg models are those summarised for example in [25].

From a Landau-Ginzburg model with the general component Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{bulk}} = & \frac{1}{2} g_{j\bar{j}} \left(\partial_+ z^j \partial_- \bar{z}^{\bar{j}} + \partial_+ \bar{z}^{\bar{j}} \partial_- z^j + i \bar{\psi}_+^{\bar{j}} \overleftrightarrow{\partial}_- \psi_+^j + i \bar{\psi}_-^{\bar{j}} \overleftrightarrow{\partial}_+ \psi_-^j \right) \\ & - \frac{1}{2} \partial_i \partial_j W(\mathbf{z}) \psi_+^i \psi_-^j - \frac{1}{2} \partial_{\bar{i}} \partial_{\bar{j}} \bar{W}(\bar{\mathbf{z}}) \bar{\psi}_-^{\bar{i}} \bar{\psi}_+^{\bar{j}} - \frac{1}{4} g^{i\bar{j}} \partial_i W(\mathbf{z}) \partial_{\bar{j}} \bar{W}(\bar{\mathbf{z}}), \end{aligned} \quad (4)$$

the plane wave theory from [17, 18] is obtained, following [11], by setting the superpotential as mentioned in the introduction to

$$W(\mathbf{z}) = -im \sum_{i=1}^4 (z^i)^2; \quad \bar{W}(\bar{\mathbf{z}}) = im \sum_{\bar{j}=1}^4 (\bar{z}^{\bar{j}})^2. \quad (5)$$

This choice gives rise to the equations of motion

$$(\partial_+ \partial_- + m^2) z^i = 0 = (\partial_+ \partial_- + m^2) \bar{z}^{\bar{i}} \quad (6)$$

for the bosons and

$$\begin{aligned} 0 = \partial_- \psi_+^j + m \bar{\psi}_-^{\bar{j}} & \quad 0 = \partial_- \bar{\psi}_+^{\bar{j}} + m \psi_-^j \\ 0 = \partial_+ \psi_-^j - m \bar{\psi}_+^{\bar{j}} & \quad 0 = \partial_+ \bar{\psi}_-^{\bar{j}} - m \psi_+^j. \end{aligned} \quad (7)$$

for the fermions.

The relation between the fermions in (4) and the standard Green-Schwarz fields S, \tilde{S} was pointed out in [11] and is given by

$$S^a = \psi_-^i \Gamma_i^{ab} \eta^b + \bar{\psi}_-^{\bar{i}} \Gamma_{\bar{i}}^{ab} \eta^{*b} \quad (8)$$

$$\tilde{S}^a = \psi_+^i \Gamma_i^{ab} \eta^b + \bar{\psi}_+^{\bar{i}} \Gamma_{\bar{i}}^{ab} \eta^{*b} \quad (9)$$

with a constant spinor η fulfilling

$$0 = \Gamma_{\bar{t}}\eta; \quad \eta^*\eta = 1; \quad \Pi\eta = -\eta^*. \quad (10)$$

The (new) Majorana type requirement $\Pi\eta = -\eta^*$ contains the real matrix

$$\Pi = \gamma^1\gamma^2\gamma^3\gamma^4 \quad (11)$$

from [18] and is consistent due to $\Pi^2 = 1$. It is chosen to determine an up to a sign unique spinor η and it correctly reproduces the equations of motion

$$\partial_+ S = m\Pi\tilde{S}; \quad \partial_- \tilde{S} = -m\Pi S \quad (12)$$

for the GS fields by starting from (7).

We will further discuss these identifications in the appendix A and close this section by stating the relation between the $\mathcal{N} = (2, 2)$ worldsheet supercharges and the spacetime supersymmetries as derived in [18]. Using the spacetime charges in the conventions of [23], the required identifications are given by

$$\frac{Q_+}{\sqrt{2p^+}} = \eta^*\tilde{Q} = -\eta\Pi\tilde{Q} \quad \frac{Q_-}{\sqrt{2p^+}} = \eta^*Q = -\eta\Pi Q \quad (13)$$

$$\frac{\bar{Q}_+}{\sqrt{2p^+}} = \eta\tilde{Q} = -\eta^*\Pi\tilde{Q} \quad \frac{\bar{Q}_-}{\sqrt{2p^+}} = \eta Q = -\eta^*\Pi Q \quad (14)$$

and we again defer a derivation to the appendix.

3 Branes in the plane wave background

In this section we briefly review the classification of (maximally) supersymmetric branes in the plane wave background from [22, 23] to explain the context of our subsequent constructions. As mentioned in the introduction, soon after the solution of the closed string theory in the plane wave background from [17] in [18], branes in this background have been studied in a significant number of papers. These discussions include various approaches, for example the use of open strings, closed string boundary states or geometric methods like probe brane settings. Starting with the papers [27], details about branes in the type II B plane wave background were derived in [29, 22, 32, 30, 23, 28, 33, 34, 35, 36, 37] and related settings are discussed for example in [38].

Following in particular the flat space treatment in [39], the (maximally) supersymmetric branes in the plane wave background have been classified in [22, 23] by using the spinor matrix

$$M = \prod_{I \in \mathcal{N}} \gamma^I. \quad (15)$$

The product is understood to span over the Neumann directions and the matrix M appears in the standard fermionic boundary conditions.

Branes of class I are characterised by

$$M\Pi M\Pi = -1 \quad (16)$$

and the maximally supersymmetric branes of this type are of structure $(r, r+2)$, $(r+2, r)$ with $r = 0, 1, 2$. Here the notation (r, s) from [29] labels the brane's orientation with respect to the $SO(4) \times SO(4)$ -background symmetry.

For class II branes one has

$$M\Pi M\Pi = 1 \quad (17)$$

and the known maximally supersymmetric branes in this class are the $(0, 0)$ instanton and the $(4, 0)$, $(0, 4)$ branes from [22, 23].

One of the main results in this paper is the construction of a maximally supersymmetric class II brane of type $(4, 4)$ with deformed fermionic boundary conditions, originating from the inclusion of boundary fermions. All our new branes will be class II branes of type (n, n) . For $n = 1, \dots, 4$ one can find alternative constructions without boundary excitations in [34]. In this case, however, there are only 4 conserved supersymmetries throughout. We defer a discussion of this setting to section 6.4.

It is worth pointing out that the inclusion of a boundary magnetic field as discussed in [35, 36] allows to construct maximally supersymmetric branes which interpolate between the class I $(2, 0)$ / $(4, 2)$ -branes and the class II $(0, 0)$ -instanton and the $(4, 0)$ brane, linking the two families in a natural way.

As mentioned in the introduction, we will begin by focussing on $\mathcal{N} = 2$ supersymmetric settings in conjunction with a preserved integrable structure following [2]. Integrability is a shared feature of all the maximally supersymmetric branes in the plane wave background¹, but is generically lost in the presence of boundary fermions with general couplings to the bulk fields. The enforcement of integrability to be discussed in the

¹This can be proven by applying the methods from [1] to be briefly mentioned in the appendix B. For the $(0, 0)$ -instanton as a particular (n, n) -brane this result will be established in due course.

next section will determine these boundary couplings up to constant parameters.

As we will conduct a study of the previously mentioned settings from an open and closed string perspective, it is important to notice that the standard light-cone gauge condition gives rise to branes of different nature in these two sectors. In the open sector the light-cone directions in the standard gauge are of Neumann type, whereas they become Dirichlet like in the closed string sector, leading therefore to instantonic boundary states, compare with [39]. As explained in [30, 23, 32], one has to apply different light-cone gauge choices in the two sectors to allow for a direct comparison. Due to this, gauge dependent quantities like the mass m appearing in (5) take on different values in the two cases. We will discuss the relation between the closed string constants m, b^i, k^i and their open string correspondents $\tilde{m}, \tilde{b}^i, \tilde{k}^i$ along the lines of [30, 23] in section 6.3.

In the following sections two directions combined to a complex variable

$$z^j = x^j + ix^{j+4} \quad j = 1, \dots, 4 \quad (18)$$

are always chosen to have the same type of boundary conditions. For later convenience we furthermore define sets $\mathcal{D}_-, \mathcal{N}_-$ containing the Dirichlet and Neumann directions ranging in $r = 1, \dots, 4$ and correspondingly $\mathcal{D}_+, \mathcal{N}_+$ with elements in $r = 5, \dots, 8$.

4 Boundary fermions: Supersymmetry and integrability

In this section we will start to construct branes in the plane wave background under the inclusion of boundary fermionic fields. In a first step, we define a suitable boundary Lagrangian and derive the corresponding boundary conditions for the bulk fields and the equations of motion for the boundary fermions.

Using these conditions, we can thereafter calculate the determining equations for the boundary fields under the requirement of conserved $\mathcal{N} = 2$ supersymmetry and integrability in the boundary theory. Further information about the integrable structure and calculational details omitted in this section can be found in the appendix B.

4.1 Boundary conditions

By mildly extending the boundary Lagrangians defined in [1, 2, 4], see also [7, 6], to include matrix valued boundary fields, we will work subsequently with the real

Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{boundary}}^{\sigma=\pi} = & \frac{i}{2} g_{j\bar{j}} \left(e^{-i\beta} \psi_-^j \bar{\psi}_+^{\bar{j}} - e^{i\beta} \psi_+^j \bar{\psi}_-^{\bar{j}} \right) - \frac{i}{2} \text{tr} \left[A \overset{\leftrightarrow}{\partial}_\tau A^\dagger \right] + B(\mathbf{z}, \bar{\mathbf{z}}) \\
& + \frac{i}{2} \text{tr} \left[\partial_{\bar{j}} F^\dagger(\bar{\mathbf{z}}) A^\dagger + \partial_{\bar{j}} G^\dagger(\bar{\mathbf{z}}) A \right] \left(\bar{\psi}_+^{\bar{j}} + e^{i\beta} \bar{\psi}_-^{\bar{j}} \right) \\
& + \frac{i}{2} \text{tr} \left[\partial_j G(\mathbf{z}) A^\dagger + \partial_j F(\mathbf{z}) A \right] \left(\psi_+^j + e^{-i\beta} \psi_-^j \right)
\end{aligned} \tag{19}$$

defined along the Neumann directions at the boundary $\sigma = \pi$. The square matrix $A = (a_{rs})$ contains the boundary fermions and F, G are matrix valued functions of the bosonic bulk fields evaluated on the boundary.

The boundary conditions along the Neumann directions deducing from the variation of (4) and (19) are found to be

$$\partial_\sigma z^j = g^{j\bar{j}} \left(\partial_{\bar{j}} B + i \text{tr} \left[\partial_{\bar{i}} \partial_{\bar{j}} F^\dagger A^\dagger + \partial_{\bar{i}} \partial_{\bar{j}} G^\dagger A \right] \bar{\theta}_+^{\bar{i}} \right) \tag{20}$$

$$\partial_\sigma \bar{z}^{\bar{i}} = g^{j\bar{j}} \left(\partial_j B + i \text{tr} \left[\partial_i \partial_j G A^\dagger + \partial_i \partial_j F A \right] \theta_+^i \right) \tag{21}$$

$$\theta_-^j = \frac{1}{2} g^{j\bar{j}} \text{tr} \left[\partial_{\bar{j}} F^\dagger A^\dagger + \partial_{\bar{j}} G^\dagger A \right] \tag{22}$$

$$\bar{\theta}_-^{\bar{j}} = \frac{1}{2} g^{j\bar{j}} \text{tr} \left[\partial_j G A^\dagger + \partial_j F A \right] \tag{23}$$

$$\partial_\tau A = \partial_{\bar{j}} F^\dagger \bar{\theta}_+^{\bar{j}} + \partial_j G \theta_+^j \tag{24}$$

$$\partial_\tau A^\dagger = \partial_{\bar{j}} G^\dagger \bar{\theta}_+^{\bar{j}} + \partial_j F \theta_+^j \tag{25}$$

which is understood to be evaluated at $\sigma = \pi$ throughout. We have furthermore used the convenient combinations

$$\begin{aligned}
\theta_+^i &= \frac{1}{2} (\psi_+^i + e^{-i\beta} \psi_-^i) & \bar{\theta}_+^{\bar{i}} &= \frac{1}{2} (\bar{\psi}_+^{\bar{i}} + e^{i\beta} \bar{\psi}_-^{\bar{i}}) \\
\theta_-^i &= \frac{1}{2} (\psi_+^i - e^{-i\beta} \psi_-^i) & \bar{\theta}_-^{\bar{i}} &= \frac{1}{2} (\bar{\psi}_+^{\bar{i}} - e^{i\beta} \bar{\psi}_-^{\bar{i}})
\end{aligned} \tag{26}$$

for the bulk fermions. By setting

$$\mathcal{L}_{\text{boundary}}^{\sigma=0} = -\mathcal{L}_{\text{boundary}}^{\sigma=\pi} \tag{27}$$

one obtains functionally the same boundary conditions at $\sigma = 0$ as derived beforehand for $\sigma = \pi$ with, however, possibly different matrices F, G at the two boundaries. Although the constraints on F and G to be derived below are also valid in the case of

different boundary fields, we will focus on the case of equal boundary conditions up to different choices for β , corresponding to brane / antibrane configurations.

Along the Dirichlet directions we will use the standard boundary conditions as for example discussed in [26]. These are in particular independent of the previously introduced boundary fermions and read explicitly

$$z^i = y_{0,\sigma}^i; \quad \bar{z}^{\bar{i}} = \bar{y}_{0,\sigma}^{\bar{i}} \quad (28)$$

$$0 = \theta_+^i; \quad 0 = \bar{\theta}_+^{\bar{i}}. \quad (29)$$

All fields are again understood to be evaluated at $\sigma = 0, \pi$.

4.2 B - type supersymmetry

As explained in section 3, we consider first of all boundary conditions by aiming at branes with two conserved B - type supersymmetries. As pointed out in [1] in a different context, the open string conservation of quantities deducing from local conserved fluxes amounts to the time independency of (in our case) the following combinations

$$Q = \bar{Q}_+ + e^{i\beta} \bar{Q}_- + \Sigma_\pi(\tau) - \Sigma_0(\tau) \quad (30)$$

$$Q^\dagger = Q_+ + e^{-i\beta} Q_- + \bar{\Sigma}_\pi(\tau) - \bar{\Sigma}_0(\tau) \quad (31)$$

with generically nonzero (*local*) contributions of boundary fields $\Sigma_\sigma(t)$ at $\sigma = \pi$ and $\sigma = 0$.

By using the supercurrents (171) and (172) presented in the appendix A, the quantities (30) and (31) are time independent in case of

$$0 = \bar{G}_+^1 + e^{i\beta} \bar{G}_-^1 \Big|_{\sigma=\pi} - \dot{\Sigma}_\pi(\tau) \quad (32)$$

$$0 = \bar{G}_+^1 + e^{i\beta} \bar{G}_-^1 \Big|_{\sigma=0} - \dot{\Sigma}_0(\tau). \quad (33)$$

Along the Dirichlet directions these conditions are trivially fulfilled with the boundary conditions (28) and (29) together with a vanishing field Σ_σ along these directions. In the case of Neumann directions with boundary conditions (20)-(25) the situation is more interesting. For a single Neumann direction the solution to (32) and (33) is discussed in detail in [15] and that treatment extends immediately to the present situation including matrix valued boundary fields. Suppressing the calculational details, we obtain the conditions

$$B = \frac{1}{2} \text{tr} [GG^\dagger + FF^\dagger] + \text{const} \quad (34)$$

$$W = ie^{-i\beta} \text{tr} [FG] + \text{const}. \quad (35)$$

The second equation (35) is understood to be valid along the Neumann directions only. For the local boundary field Σ_π we furthermore have

$$\Sigma_\pi(\tau) = -2g_{j\bar{j}}\bar{\theta}^{\bar{j}}_- z^j + \text{tr} \left[(z^j \partial_j F - F) A + (z^j \partial_j G - G) A^\dagger \right], \quad (36)$$

compare again with [15].

4.3 Integrability

Although arbitrary boundary fields obeying (34) and (35) already give rise to $\mathcal{N} = 2$ supersymmetrical settings, we are here interested in the more restricted case of *integrable* boundary conditions, that is, branes which also respect the integrable structure present in the bulk theory. As explained in section 3, all known maximally supersymmetric branes in the plane wave theory are actually also integrable. By the inclusion of boundary fields as in (19), this integrability conservation is a priori no longer guaranteed and leads, if enforced, to further constraints on admissible boundary conditions.

In this section we will give the explicit expression of two higher spin bulk currents and state the conditions for their conservation in the presence of boundaries. This conservation gives strong evidence for the integrability of the boundary theory. To further underpin the actual presence of such a structure one might use the explicit mode expansions to be derived in the next section and compare them with the requirements derived in [1] for integrable boundary field theories. We will briefly comment on this in the appendix B.

Local conserved higher spin currents for the massive Ising model were written down in [40]. Here we focus on combinations which, for a single Neumann direction, appear as limiting cases of the first nontrivial higher spin currents in the $\mathcal{N} = 2$ sine-Gordon model. We defer a more detailed discussion of this point to the appendix B where we also supply the infinite series of conserved fluxes from [40].

In manifestly real form the currents of present interest are given by

$$T_4 = g_{i\bar{i}} \left(\partial_+^2 \bar{z}^{\bar{i}} \partial_+^2 z^i + \frac{i}{2} \partial_+ \bar{\psi}^{\bar{i}} \partial_+^2 \psi^i - \frac{i}{2} \partial_+^2 \bar{\psi}^{\bar{i}} \partial_+ \psi^i \right) \quad (37)$$

$$\theta_2 = g_{i\bar{i}} \left(-m^2 \partial_+ \bar{z}^{\bar{i}} \partial_+ z^i - \frac{im^2}{2} \bar{\psi}^{\bar{i}} \partial_+ \psi^i + \frac{im^2}{2} \partial_+ \bar{\psi}^{\bar{i}} \psi^i \right) \quad (38)$$

and

$$\bar{T}_4 = g_{i\bar{i}} \left(\partial_-^2 \bar{z}^{\bar{i}} \partial_-^2 z^i + \frac{i}{2} \partial_- \bar{\psi}^{\bar{i}} \partial_-^2 \psi^i - \frac{i}{2} \partial_-^2 \bar{\psi}^{\bar{i}} \partial_- \psi^i \right) \quad (39)$$

$$\bar{\theta}_2 = g_{i\bar{i}} \left(-m^2 \partial_- \bar{z}^{\bar{i}} \partial_- z^i - \frac{im^2}{2} \bar{\psi}^{\bar{i}} \partial_- \psi^i + \frac{im^2}{2} \partial_- \bar{\psi}^{\bar{i}} \psi^i \right) \quad (40)$$

and fulfil on-shell

$$\partial_- T_4 = \partial_+ \theta_2; \quad \partial_+ \bar{T}_4 = \partial_- \bar{\theta}_2. \quad (41)$$

In the bulk theory both fluxes give rise to conserved spin 3 operators. The conservation of a suitable combination of the previous operators in the presence of boundaries is discussed in the appendix. There the conditions for integrability are found to be

$$\begin{aligned} \partial_i \partial_j \partial_k B &= 0 & \partial_i \partial_{\bar{j}} \partial_{\bar{k}} B &= 0 \\ \partial_i \partial_j \partial_{\bar{k}} B &= 0 & \partial_i \partial_{\bar{j}} \partial_k B &= 0 \end{aligned} \quad (42)$$

for the boundary potential and

$$\begin{aligned} 0 &= \text{tr} (\partial_i \partial_j G A^\dagger + \partial_i \partial_j F A) \\ 0 &= \text{tr} (\partial_i \partial_{\bar{j}} G^\dagger A + \partial_i \partial_{\bar{j}} F^\dagger A^\dagger) \end{aligned} \quad (43)$$

for the matrices F and G .

Having presented the conditions for $\mathcal{N} = 2$ supersymmetry (34), (35) in the last section and for integrability in (42) and (43), it is now straightforward to write down the corresponding solutions. They are given by

$$F = A_i z^i + C \quad G = B_i z^i + D \quad (44)$$

along the Neumann directions with

$$\text{tr} (A_i B_j) = -e^{i\beta} \tilde{m} \delta_{ij} \quad \text{tr} (A_i D + B_i C) = 0. \quad (45)$$

The resulting boundary potential becomes up to an irrelevant constant

$$B(\mathbf{z}, \bar{\mathbf{z}}) = \frac{1}{2} \text{tr} (A_i A_j^\dagger + B_i B_j^\dagger) z^i \bar{z}^j + \text{tr} (A_i C^\dagger + B_i D^\dagger) z^i + \text{tr} (C A_i^\dagger + D B_i^\dagger) \bar{z}^i, \quad (46)$$

again extending only along the Neumann directions.

5 The open string with boundary fermions

In this section we present a detailed discussion of (n, n) -branes with $n = 0, \dots, 4$ from an open string point of view by enforcing Neumann boundary conditions as introduced in the last section. Using the equations of motion for the boundary fermions we can eliminate these extra fields from the remaining boundary conditions. Although the resulting boundary conditions on the fermionic bulk fields differ clearly from the standard settings, the corresponding solutions can be found and quantised by standard methods.

As stated in the introduction, the boundary fermions can be expressed in terms of the bulk fields restricted to the boundary without including additional degrees of freedom. We explain in detail how this solutions reproduces the expected anticommutators of the boundary fermions in the quantum theory. The section closes with a derivation of the $\mathcal{N} = 2$ superalgebra of the boundary theory. These results will be needed in the discussion of the open-closed duality in section 7.

For the boundary fields appearing in the Neumann directions we will work with a particular solution of type (44) given by

$$F = \text{diag}(A^{\hat{i}} z^{\hat{i}} + C^i); \quad G = \text{diag}(B^{\hat{i}} z^{\hat{i}} + D^i) \quad (47)$$

with no sum over hatted indices. The solution (47) allows us to treat the fields along any complex direction $z^{\hat{i}}$ separately and construct (n, n) -type branes for all n in a single approach.

We will consider only strings spanning between branes with the same type of boundary fields and restrict the parameter β appearing in (19) to the values 0 and π corresponding to brane or antibrane settings. The latter will again be needed in section 7. The more general situation of $\beta \in (0, \pi)$ can be dealt with with the methods explained in [35] in the context of boundary magnetic fields in the plane wave background.

For future reference we note here the most general solutions to the equations of motion (6) and (7) which read in a real basis

$$0 = (\partial_+ \partial_- + \tilde{m}^2) X^s \quad (48)$$

for the bosons with $s = 1, \dots, 8$ and

$$\partial_- \psi_+^t = -\tilde{m} \psi_-^t \quad \partial_+ \psi_-^t = \tilde{m} \psi_+^t \quad (49)$$

$$\partial_- \psi_+^{t+4} = +\tilde{m} \psi_-^{t+4} \quad \partial_+ \psi_-^{t+4} = -\tilde{m} \psi_+^{t+4} \quad (50)$$

for the fermions with $t = 1, \dots, 4$.

The fermionic fields along the $s = 5, \dots, 8$ directions are obtained from those along the $s = 1, \dots, 4$ directions by interchanging $\tilde{m} \leftrightarrow -\tilde{m}$, reflecting the different eigenvalues of the matrix Π introduced in section 3.

Following [23], the most general solutions to (48)-(50) are given by

$$\begin{aligned} X^s(\tau, \sigma) = & C^s \sin(\tilde{m}\tau) + \tilde{C}^s \cos(\tilde{m}\tau) + D^s \cosh(\tilde{m}\sigma) + \tilde{D}^s \sinh(\tilde{m}\sigma) \\ & + i \sum_{\substack{n, \omega_n \neq 0 \\ \omega_n^2 = n^2 + \tilde{m}^2}} \frac{1}{\omega_n} (a_n^s e^{-i(\omega_n\tau - n\sigma)} + \tilde{a}_n^s e^{-i(\omega_n\tau + n\sigma)}) \end{aligned} \quad (51)$$

and

$$\begin{aligned} \psi_+^t(\tau, \sigma) = & -\phi^t \sin(\tilde{m}\tau) + \tilde{\phi}^t \cos(\tilde{m}\tau) + \tilde{\psi}^t \cosh(\tilde{m}\sigma) + \psi^t \sinh(\tilde{m}\sigma) \\ & + \sum_{\substack{n, \omega_n \neq 0 \\ \omega_n^2 = n^2 + \tilde{m}^2}} c_n \left(\tilde{\psi}_n^t e^{-i(\omega_n\tau + n\sigma)} - \frac{i}{\tilde{m}} (\omega_n - n) \psi_n^t e^{-i(\omega_n\tau - n\sigma)} \right) \end{aligned} \quad (52)$$

$$\begin{aligned} \psi_-^t(\tau, \sigma) = & \phi^t \cos(\tilde{m}\tau) + \tilde{\phi}^t \sin(\tilde{m}\tau) + \tilde{\psi}^t \sinh(\tilde{m}\sigma) + \psi^t \cosh(\tilde{m}\sigma) \\ & + \sum_{\substack{n, \omega_n \neq 0 \\ \omega_n^2 = n^2 + \tilde{m}^2}} c_n \left(\psi_n^t e^{-i(\omega_n\tau - n\sigma)} + \frac{i}{\tilde{m}} (\omega_n - n) \tilde{\psi}_n^t e^{-i(\omega_n\tau + n\sigma)} \right) \end{aligned} \quad (53)$$

with

$$c_n = \frac{\tilde{m}}{\sqrt{2\omega_n(\omega_n - n)}}. \quad (54)$$

5.1 Dirichlet directions

In this section we will consider the bulk fields spanning along a Dirichlet direction with boundary conditions

$$X^s(\tau, \sigma = 0) = y_0^s; \quad X^s(\tau, \sigma = \pi) = y_\pi^s \quad (55)$$

and

$$0 = (\psi_+^s + \rho \psi_-^s)(\tau, \sigma = 0, \pi). \quad (56)$$

Here $\rho = \pm 1$ distinguishes as usual between the brane / antibrane cases. Our discussion proceeds in this part along the lines of the $(0, 0)$ -instanton construction from [23], but

differs mildly in the fermionic sector due to our choice of LG-fermions as discussed in section 2.

From (51), the boundary conditions (55) and (56) lead to the bosonic mode expansion

$$X^s(\tau, \sigma) = x_0^s \cosh(\tilde{m}\sigma) + \frac{x_\pi^s - x_0^s \cosh(\tilde{m}\pi)}{\sinh(\tilde{m}\pi)} \sinh(\tilde{m}\sigma) - \sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{\omega_n} e^{-i\omega_n \tau} a_n^s \sin(n\sigma) \quad (57)$$

with $\omega_n = \text{sgn}(n)\sqrt{n^2 + \tilde{m}^2}$, compare for example with [23].

For the fermions spanning between a brane-brane configuration we deduce for $t \in \mathcal{D}_-$

$$\psi_+^t(\tau, \sigma) = -\psi^t e^{-\tilde{m}\sigma} + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left(\tilde{\psi}_n^t e^{-i(\omega_n \tau + n\sigma)} - i \frac{\omega_n - n}{\tilde{m}} \psi_n^t e^{-i(\omega_n \tau - n\sigma)} \right) \quad (58)$$

$$\psi_-^t(\tau, \sigma) = \psi^t e^{-\tilde{m}\sigma} + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left(\psi_n^t e^{-i(\omega_n \tau - n\sigma)} + i \frac{\omega_n - n}{\tilde{m}} \tilde{\psi}_n^t e^{-i(\omega_n \tau + n\sigma)} \right) \quad (59)$$

with the identifications

$$\tilde{\psi}_n^t = -\frac{n - i\tilde{m}}{\omega_n} \psi_n^t. \quad (60)$$

As explained before, the solutions along the directions $t \in \mathcal{D}_+$ are obtained from (58) and (59) by using $\tilde{m} \rightarrow -\tilde{m}$.

The fermionic fields spanning between a brane/antibrane combination have the same structure as presented in (58) and (59). In this case, however, the zero modes ψ^t are absent and the nonzero modings have to fulfil either

$$e^{2\pi i n} = -\frac{n - i\tilde{m}}{n + i\tilde{m}} \quad \text{or} \quad e^{2\pi i n} = -\frac{n + i\tilde{m}}{n - i\tilde{m}}; \quad n \neq 0 \quad (61)$$

depending on whether $t \in \mathcal{D}_-$ or $t \in \mathcal{D}_+$, compare again with the discussion of the (0,0) instanton in [23].

5.1.1 Quantisation

By requiring the standard canonical commutators as summarised in the appendix C we obtain the commutation relations for the modes introduced in the last section to

$$[a_m^i, a_n^j] = \omega_m \delta^{ij} \delta_{m+n} \quad (62)$$

$$\{\psi_m^r, \psi_n^s\} = \delta^{rs} \delta_{m+n} \quad (63)$$

$$\{\psi^r, \psi^s\} = \frac{2\pi\tilde{m}}{1 - e^{-2\pi\tilde{m}}} \delta^{rs} = \frac{\pi\tilde{m}e^{\pi\tilde{m}}}{\sinh(\pi\tilde{m})} \delta^{rs}. \quad (64)$$

The anticommutators are written down for parameters r, s ranging in \mathcal{D}_- . Some details of the derivations, in particular of (64), can be found in the appendix C.

5.2 Neumann directions

In this part we will consider the mode expansions for the new Neumann type boundary conditions including contributions of the boundary Lagrangian as discussed above. We will work with the boundary fields presented in equation (47) whose parameters fulfil

$$A^{\hat{i}} B^{\hat{i}} = -e^{i\beta} \tilde{m}; \quad A^{\hat{i}} D^{\hat{i}} + C^{\hat{i}} B^{\hat{i}} = 0 \quad (65)$$

to obey (35), ensuring in particular the conservation of a $\mathcal{N} = 2$ supersymmetry structure. As before, there is no sum over hatted indices. Using (47), the boundary potential B from (34) takes on the structure

$$B(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i \in \mathcal{N}} \left(\tilde{b}^i z^i \bar{z}^{\bar{i}} + \tilde{k}^i \bar{z}^{\bar{i}} + \bar{\tilde{k}}^{\bar{i}} z^i \right) + \text{const} \quad (66)$$

by using the convenient combinations

$$\tilde{b}^i = \tilde{b}^I = \tilde{b}^{I+4} = \frac{A^{\hat{i}} \bar{A}^{\hat{i}} + B^{\hat{i}} \bar{B}^{\hat{i}}}{2} \quad (67)$$

$$\tilde{k}^i = \frac{C^{\hat{i}} \bar{A}^{\hat{i}} + D^{\hat{i}} \bar{B}^{\hat{i}}}{2}; \quad \bar{\tilde{k}}^{\bar{i}} = \frac{\bar{C}^{\hat{i}} A^{\hat{i}} + \bar{D}^{\hat{i}} B^{\hat{i}}}{2}. \quad (68)$$

With (65) we furthermore have

$$A^{\hat{i}} \bar{A}^{\hat{i}} = \tilde{b}^i \pm \sqrt{(\tilde{b}^i)^2 - \tilde{m}^2}; \quad k^i = \pm \frac{C^{\hat{i}}}{A^{\hat{i}}} \sqrt{(\tilde{b}^i)^2 - \tilde{m}^2} \quad (69)$$

for

$$0 < \tilde{m} \leq \tilde{b}^i \quad \text{and} \quad i \in \mathcal{N}_-. \quad (70)$$

In this section we will assume throughout $m < b^i$ and comment on the limiting cases $b^i = m$ and their relation in the bosonic sector to previously known branes later on in section 6.4.

From (20)-(23), the boundary conditions at $\sigma = 0, \pi$ become

$$\partial_\sigma X^I = \tilde{b}^{\hat{I}} X^{\hat{I}} + \tilde{k}^I \quad (71)$$

for the bosons with $I \in \mathcal{N}$. For the fermionic boundary conditions we use the boundary equations of motion (24) and (25) to eliminate the boundary fermions from (22) and (23) and derive

$$\partial_\tau (\psi_+^I - \rho \psi_-^I) = (\tilde{b}^{\hat{I}} - \rho \tilde{m}) (\psi_+^{\hat{I}} + \rho \psi_-^{\hat{I}}) \quad (72)$$

$$\partial_\tau (\psi_+^{I+4} - \rho \psi_-^{I+4}) = (\tilde{b}^{\hat{I}} + \rho \tilde{m}) (\psi_+^{\hat{I}+4} + \rho \psi_-^{\hat{I}+4}) \quad (73)$$

for the fermionic bulk fields with $\sigma = 0, \pi$ and $I \in \mathcal{N}_-$. Both cases are formulated in a real basis and the parameter ρ distinguishes as before between the brane / antibrane boundary conditions.

Using the general solution (51) together with the boundary conditions (71) the bosonic mode expansions along the Neumann directions are found to be

$$\begin{aligned} X^I(\tau, \sigma) = & N^I \cosh(\tilde{m}\sigma) + \tilde{N}^I \sinh(\tilde{m}\sigma) + P^{\hat{I}} e^{\sqrt{(\tilde{b}^{\hat{I}})^2 - \tilde{m}^2} \tau} e^{\tilde{b}^{\hat{I}} \sigma} + Q^{\hat{I}} e^{-\sqrt{(\tilde{b}^{\hat{I}})^2 - \tilde{m}^2} \tau} e^{\tilde{b}^{\hat{I}} \sigma} \\ & + \frac{i}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{\omega_n} (a_n^I e^{-i(\omega_n \tau - n\sigma)} + \tilde{a}_n^I e^{-i(\omega_n \tau + n\sigma)}) \end{aligned} \quad (74)$$

with

$$\tilde{a}_n^I = \frac{n + i\tilde{b}^{\hat{I}}}{n - i\tilde{b}^{\hat{I}}} a_n^{\hat{I}} \quad (75)$$

and

$$N^I = \frac{\tilde{b}^{\hat{I}} \cosh \frac{\tilde{m}\pi}{2} - \tilde{m} \sinh \frac{\tilde{m}\pi}{2}}{(\tilde{m}^2 - (\tilde{b}^{\hat{I}})^2) \cosh \frac{\tilde{m}\pi}{2}} \tilde{k}^{\hat{I}} \quad (76)$$

$$\tilde{N}^I = \frac{\tilde{m} \cosh \frac{\tilde{m}\pi}{2} - \tilde{b}^{\hat{I}} \sinh \frac{\tilde{m}\pi}{2}}{(\tilde{m}^2 - (\tilde{b}^{\hat{I}})^2) \cosh \frac{\tilde{m}\pi}{2}} \tilde{k}^{\hat{I}}. \quad (77)$$

The special modes P^I, Q^I with a time dependency proportional to $e^{\pm\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2}\tau}$ are of the same type as those appearing in [33, 35] in the treatment of open strings in the plane wave background under the inclusion of a nontrivial \mathcal{F}^{IJ} -field. They play a crucial rôle in the quantisation to be discussed in the next section.

For the fermions spanning between a brane/brane pair with $\rho = 1$ we obtain from (52),(53) and the boundary conditions (72) the solutions

$$\begin{aligned}\psi_+^I(\tau, \sigma) = & -\psi^I e^{-\tilde{m}\sigma} + e^{-\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2}\tau} e^{\tilde{b}^I \sigma} \chi^{\hat{I}} + e^{\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2}\tau} e^{\tilde{b}^I \sigma} \frac{\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} + \tilde{b}^{\hat{I}}}{\tilde{m}} \tilde{\chi}^{\hat{I}} \\ & + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left(\tilde{\psi}_n^I e^{-i(\omega_n \tau + n\sigma)} - i \frac{\omega_n - n}{m} \psi_n^I e^{-i(\omega_n \tau - n\sigma)} \right)\end{aligned}\quad (78)$$

$$\begin{aligned}\psi_-^I(\tau, \sigma) = & \psi^I e^{-m\sigma} + e^{\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2}\tau} e^{\tilde{b}^I \sigma} \tilde{\chi}^{\hat{I}} + e^{-\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2}\tau} e^{\tilde{b}^I \sigma} \frac{\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} + \tilde{b}^{\hat{I}}}{\tilde{m}} \chi^{\hat{I}} \\ & + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left(\psi_n^I e^{-i(\omega_n \tau - n\sigma)} + i \frac{\omega_n - n}{m} \tilde{\psi}_n^I e^{-i(\omega_n \tau + n\sigma)} \right)\end{aligned}\quad (79)$$

with

$$\tilde{\psi}_n^I = \frac{\omega_n}{n + i\tilde{m}} \frac{n + i\tilde{b}^{\hat{I}}}{n - i\tilde{b}^{\hat{I}}} \psi_n^{\hat{I}} \quad (80)$$

and $I \in \mathcal{N}_-$. The modes χ^I and $\tilde{\chi}^I$ correspond to the bosonic operators P^I, Q^I , compare for example with [35]. As described there, the terms in (78) and (79) containing these special modes fulfil the conditions (72), (73) for all σ and not only on the boundary.

The remaining fermionic solutions along the $I \in \mathcal{N}_+$ directions are again deduced by sending $\tilde{m} \rightarrow -\tilde{m}$ in (78) and (79). In particular, one obtains the mode identifications for the nonzero modes in this case to

$$\tilde{\psi}_n^{I+4} = \frac{\omega_n}{n - i\tilde{m}} \frac{n + i\tilde{b}^{\hat{I}}}{n - i\tilde{b}^{\hat{I}}} \psi_n^{\hat{I}+4}. \quad (81)$$

As for the Dirichlet directions, the mode expansion for strings stretching between a brane / antibrane pair deduces from (78) and (79) by dropping the zero modes ψ^I , but retaining the special modes χ^I and $\tilde{\chi}^I$. Furthermore, the moding for the nonzero modes again has to fulfil either

$$e^{2\pi i n} = -\frac{n - i\tilde{m}}{n + i\tilde{m}} \quad \text{or} \quad e^{2\pi i n} = -\frac{n + i\tilde{m}}{n - i\tilde{m}} \quad (82)$$

depending on whether $I \in \mathcal{N}_-$ or $I \in \mathcal{N}_+$.

5.2.1 Quantisation

The standard canonical conditions (186)-(191) lead in the Neumann case to the following commutators. For the bosons we obtain

$$[P^I, Q^J] = \delta^{IJ} \frac{2\pi i \tilde{b}^{\hat{I}}}{\sqrt{(\tilde{b}^{\hat{I}})^2 - \tilde{m}^2}} \frac{1}{1 - e^{2\pi \tilde{b}^{\hat{I}}}} \quad (83)$$

$$[a_m^I, a_n^J] = \omega_m \delta^{IJ} \delta_{m+n} \quad (84)$$

whereas for the fermions

$$\{\psi_m^I, \psi_n^J\} = \delta^{IJ} \delta_{m+n} \quad (85)$$

$$\{\psi^I, \psi^J\} = -\frac{2\pi \tilde{m} \delta^{\hat{I}J}}{1 - e^{-2\pi \tilde{m}}} \frac{\tilde{m} - \tilde{b}^{\hat{I}}}{\tilde{m} + \tilde{b}^{\hat{I}}} \quad (86)$$

$$\{\chi^I, \tilde{\chi}^J\} = \frac{2\pi \tilde{b}^{\hat{I}} \delta^{\hat{I}J}}{1 - e^{2\pi \tilde{b}^{\hat{I}}}} \frac{\sqrt{(\tilde{b}^{\hat{I}})^2 - \tilde{m}^2} - \tilde{b}^{\hat{I}}}{\tilde{b}^{\hat{I}} + \tilde{m}}. \quad (87)$$

The fermionic relations are again formulated for $I, J \in \mathcal{N}_-$ only. Some details of the derivations are presented in the appendix C.

5.3 Boundary fermions

In the last section the boundary fermionic fields were eliminated from the remaining boundary conditions by using their equations of motion. In this section we reconsider this situation and present the explicit solution for the boundary fermions as a suitable combination of fermionic bulk fields evaluated on the boundary.

For our choice of diagonal matrices F, Q all non-diagonal elements of A, A^\dagger in (19) decouple from the remaining fields and we can therefore concentrate on the diagonal components. For these elements we have to solve the equations of motion (24) and (25) by using (47). For notational simplicity we will write down only expressions for fermions corresponding to the z^1 direction and suppress for this case irrelevant indices.

By using (72) and (73) in the equations of motion (24) and (25) we obtain the boundary fermions to

$$a(t) = a_0 + \frac{\bar{A} + B}{2(\tilde{b} - \tilde{m})} (\psi_+^1 - \psi_-^1) - i \frac{\bar{A} - B}{2(\tilde{b} + \tilde{m})} (\psi_+^5 - \psi_-^5) \quad (88)$$

$$\bar{a}(t) = \bar{a}_0 + \frac{A + \bar{B}}{2(\tilde{b} - \tilde{m})} (\psi_+^1 - \psi_-^1) + i \frac{A - \bar{B}}{2(\tilde{b} + \tilde{m})} (\psi_+^5 - \psi_-^5) \quad (89)$$

with constant fermions a_0, \bar{a}_0 . Using so far only to the differentiated boundary conditions (72) and (73), we have to test whether there are additional constraints on these extra fermions. From the (undifferentiated) conditions (22) and (23) we obtain

$$0 = B\bar{a}_0 + Aa_0 \quad (90)$$

which amounts to $a_0 = \bar{a}_0 = 0$ by using the explicit expressions for A and B from (65) and (69) with $\tilde{b} > \tilde{m}$. For our solution (47) all boundary fermions in (19) therefore either decouple from the remaining fields or are expressible in terms of bulk functions restricted to the boundary.

For consistency of the last result, the fermionic anticommutation relations for the bulk fields derived in section 5.2.1 should reproduce the expected anticommutators for the boundary fermionic fields $a(t)$ and $\bar{a}(t)$. To determine these relations we have to evaluate expressions like

$$(\star) = \{\psi_+^1(\tau, \sigma) - \psi_-^1(\tau, \sigma), \psi_+^1(\tau, \bar{\sigma}) - \psi_-^1(\tau, \bar{\sigma})\} \quad (91)$$

at the boundaries. This is, different to the bulk, relatively subtle due to potential divergencies. Using (86) and (87) we obtain

$$(\star) = -\frac{8\pi\tilde{m}}{1 - e^{-2\pi\tilde{m}}} \frac{\tilde{m} - \tilde{b}}{\tilde{m} + \tilde{b}} e^{-\tilde{m}(\sigma+\bar{\sigma})} - \frac{8\pi\tilde{b}}{1 - e^{2\pi\tilde{b}}} \frac{\tilde{m} - \tilde{b}}{\tilde{m} + \tilde{b}} e^{\tilde{b}(\sigma+\bar{\sigma})} \quad (92)$$

$$+ 2 \sum_{n \neq 0} \left(e^{in(\sigma-\bar{\sigma})} - \frac{n + i\tilde{m}}{n - i\tilde{m}} \frac{n - i\tilde{b}}{n + i\tilde{b}} e^{in(\sigma+\bar{\sigma})} \right). \quad (93)$$

After setting one of the arguments $\sigma, \bar{\sigma}$ equal to the boundary values 0 or π we have for the infinite sum

$$4i(\tilde{b} - \tilde{m}) \sum_{n \neq 0} \frac{ne^{in(\sigma+\bar{\sigma})}}{(n - i\tilde{m})(n + i\tilde{b})} = -4i(\tilde{b} - \tilde{m}) \oint_{\mathcal{C}} dz \frac{e^{iz(\sigma+\bar{\sigma})}}{1 - e^{2\pi iz}} \frac{z}{(z - i\tilde{m})(z + i\tilde{b})} \quad (94)$$

where \mathcal{C} is a contour running infinitesimally above and below the real axis, compare with [23]. By closing the contours the residues cancel out with the first terms in (\star) and we finally obtain

$$\{\psi_+^1(\tau, \sigma) - \psi_-^1(\tau, \sigma), \psi_+^1(\tau, \bar{\sigma}) - \psi_-^1(\tau, \bar{\sigma})\} = \pm 4\pi(\tilde{b} - \tilde{m}) \quad (95)$$

$$\{\psi_+^5(\tau, \sigma) - \psi_-^5(\tau, \sigma), \psi_+^5(\tau, \bar{\sigma}) - \psi_-^5(\tau, \bar{\sigma})\} = \pm 4\pi(\tilde{b} + \tilde{m}) \quad (96)$$

at $\sigma = \bar{\sigma} = 0$ and $\sigma = \bar{\sigma} = \pi$, respectively. Using (88) and (89) this leads to the anticommutators

$$\{a(t), a(t)\} = 0 \quad (97)$$

$$\{\bar{a}(t), \bar{a}(t)\} = 0 \quad (98)$$

$$\{a(t), \bar{a}(t)\} = \pm 4\pi \quad (99)$$

which is the expected result. The signs originate here in different overall signs appearing in the boundary Lagrangian (19) at the two boundaries.

5.4 The $\mathcal{N} = 2$ superalgebra

In this final section of our open string treatment we will determine the Hamiltonians of the previously discussed configurations and for the brane-brane situation also the resulting $\mathcal{N} = 2$ supercharges. The expressions for the Hamiltonians will be put to use in section 7.

The conserved supercharges are calculated from the equations (30) and (31) established in section 4.2 whereas the Hamiltonians deduce from the following (closed string) conserved fluxes

$$T_2 = g_{j\bar{j}} \left(\partial_+ \bar{z}^{\bar{j}} \partial_+ z^j + \frac{i}{2} \bar{\psi}_+^{\bar{j}} \overleftrightarrow{\partial}_+ \psi_+ \right) \quad \theta_0 = g_{j\bar{j}} \left(-m^2 \bar{z}^{\bar{j}} z^j - \frac{i}{2} \bar{\psi}_+^{\bar{j}} \overleftrightarrow{\partial}_- \psi_+^j \right) \quad (100)$$

$$\bar{T}_2 = g_{j\bar{j}} \left(\partial_- \bar{z}^{\bar{j}} \partial_- z^j + \frac{i}{2} \bar{\psi}_-^{\bar{j}} \overleftrightarrow{\partial}_- \psi_- \right) \quad \bar{\theta}_0 = g_{j\bar{j}} \left(-m^2 \bar{z}^{\bar{j}} z^j - \frac{i}{2} \bar{\psi}_-^{\bar{j}} \overleftrightarrow{\partial}_+ \psi_-^j \right) \quad (101)$$

which fulfil on-shell

$$\partial_- T_2 = \partial_+ \theta_0; \quad \partial_+ \bar{T}_2 = \partial_- \bar{\theta}_0. \quad (102)$$

5.4.1 Dirichlet directions

Using the fluxes (100) and (101) the open string Hamiltonian along the Dirichlet directions for a brane/brane configuration becomes with the mode expansions (57)-(59) in the overall normalisation explained in detail in [23]

$$\begin{aligned} \frac{X^+}{2\pi} H^{\text{open}} &= \frac{\tilde{m}}{2 \sinh(\tilde{m}\pi)} \sum_{a \in \mathcal{D}} (\cosh(\tilde{m}\pi) (x_0^a x_0^a + x_\pi^a x_\pi^a) - 2 x_0^a x_\pi^a) \\ &\quad + 2\pi \sum_{\substack{n > 0 \\ a \in \mathcal{D}}} (a_{-n}^a a_n^a + \omega_n \psi_{-n}^a \psi_n^a), \end{aligned} \quad (103)$$

where the summation index a is understood to range over all Dirichlet directions.

The Hamiltonian for the brane/antibrane configuration has the same structure with a fermionic nonzero moding as given in (61). In this case there is also an overall normal ordering constant which will be implicitly determined in section 7.

The contribution to the overall $\mathcal{N} = 2$ supercharges for a brane/brane configuration with $\rho = 1$ becomes

$$Q = 2 \sum_{a \in \mathcal{D}_-} ((\psi^a - i\psi^{a+4}) (x_0^a + ix_0^{a+4}) - (e^{-\tilde{m}\pi}\psi^a - ie^{\tilde{m}\pi}\psi^{a+4}) (x_\pi^a + ix_\pi^{a+4})) \\ + 2\pi\sqrt{2} \sum_{\substack{n \neq 0 \\ a \in \mathcal{D}_-}} c_n \left[\left(1 - i\frac{\omega_n - n}{\tilde{m}}\right) \psi_n^a - i \left(1 + i\frac{\omega_n - n}{\tilde{m}}\right) \psi_n^{a+4} \right] (a_{-n}^a + ia_{-n}^{a+4}) \quad (104)$$

with the corresponding complex conjugated expression for Q^\dagger .

5.4.2 Neumann directions

Along the Neumann directions the fluxes (100) and (101) require the inclusion of boundary currents in the open string sector as discussed in section 4.2 for the supercharges and in the appendix B for the higher spin currents of the integrable structure. In the present case, the local boundary field has the form

$$\Sigma_\pi^{(1)} = 2 \left(B(z, \bar{z}) + ig_{j\bar{j}} \left(\bar{\theta}_-^j \theta_+^j - \bar{\theta}_+^j \theta_-^j \right) \right) \quad (105)$$

and the suitable normalised Hamiltonian becomes for open strings stretching between a brane / brane pair

$$\frac{X^+}{2\pi} H^{\text{open}} = H_0 + \sum_{I \in \mathcal{N}} \left[\frac{\tilde{m}^2 - (\tilde{b}^I)^2}{\tilde{b}^I} \left(e^{2\pi\tilde{b}^I} - 1 \right) Q^I P^I \right] + 2\pi \sum_{\substack{n > 0 \\ I \in \mathcal{N}}} (a_{-n}^I a_n^I + \omega_n \psi_{-n}^I \psi_n^I) \\ + i \sum_{I \in \mathcal{N}_-} \left[(e^{2\pi\tilde{b}^I} - 1) \frac{((\tilde{b}^I)^2 - \tilde{m}^2)^{\frac{3}{2}}}{\tilde{b}^I \tilde{m}^2} \left(\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} + \tilde{b}^I \right) \left(\frac{\chi^I \tilde{\chi}^I}{\tilde{b}^I - \tilde{m}} + \frac{\chi^{I+4} \tilde{\chi}^{I+4}}{\tilde{b}^I + \tilde{m}} \right) \right] \quad (106)$$

with

$$\begin{aligned}
2H_0 = & \sum_{I \in \mathcal{N}} \left[\left(\tilde{m} \cosh(\tilde{m}\pi) - \tilde{b}^I \sinh(\tilde{m}\pi) \right) \sinh(\tilde{m}\pi) \left(N^I N^I + \tilde{N}^I \tilde{N}^I \right) \right. \\
& + 2 \left(\tilde{m} \sinh(\tilde{m}\pi) - \tilde{b}^I \cosh(\tilde{m}\pi) \right) \sinh(\tilde{m}\pi) N^I \tilde{N}^I \\
& \left. - 2\tilde{k}^I \left(N^I (\cosh(\tilde{m}\pi) - 1) + \tilde{N}^I \sinh(\tilde{m}\pi) \right) \right] \quad (107)
\end{aligned}$$

as contribution from the bosonic zero modes. Using (76) and (77) this simplifies to

$$H_0 = \tilde{m} \sum_{I \in \mathcal{N}} \frac{\tanh \frac{\tilde{m}\pi}{2}}{(\tilde{b}^I)^2 - \tilde{m}^2} \tilde{k}^I \tilde{k}^I. \quad (108)$$

The Hamiltonian (106) is already presented in its normal ordered form by implicitly defining P^I and $\tilde{\chi}^I$ as annihilation operators for the special zero-modes. With these choices the corresponding normal ordering constants cancel.

The Hamiltonian for open strings in between a brane - antibrane pair also has the structure (106). In that case, however, the fermionic moding has to fulfil (61) and there also appears a nonzero normal ordering constant. It solely originates from the nonzero modes and takes on the same value as in the previously discussed Dirichlet case.

The contributions to the supercharge in the case of strings in between two branes with $\rho = 0$ is finally obtained to

$$\begin{aligned}
Q = & \sum_{I \in \mathcal{N}_-} \left[2 \left(\tilde{k}^I + i\tilde{k}^{I+4} \right) \left(\frac{e^{-\tilde{m}\pi} - 1}{\tilde{b}^I - \tilde{m}} \psi^I - i \frac{e^{\tilde{m}\pi} - 1}{\tilde{b}^I + \tilde{m}} \psi^{I+4} \right) \right. \\
& + \frac{e^{2\pi\tilde{b}^I} - 1}{\tilde{m}\tilde{b}^I} \sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} \left(\sqrt{\tilde{b}^I + \tilde{m}} + \sqrt{\tilde{b}^I - \tilde{m}} \right) \\
& \times \left(\sqrt{\tilde{b}^I + \tilde{m}} \chi^I + i\sqrt{\tilde{b}^I - \tilde{m}} \chi^{I+4} \right) (P^I + iP^{I+4}) \\
& - \frac{e^{2\pi\tilde{b}^I} - 1}{\tilde{m}\tilde{b}^I} \sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} \left(\sqrt{\tilde{b}^I + \tilde{m}} + \sqrt{\tilde{b}^I - \tilde{m}} \right) \\
& \times \left(\sqrt{\tilde{b}^I + \tilde{m}} \tilde{\chi}^I + i\sqrt{\tilde{b}^I - \tilde{m}} \tilde{\chi}^{I+4} \right) (Q^I + iQ^{I+4}) \Big] \\
& + 2\pi\sqrt{2} \sum_{\substack{n \neq 0 \\ I \in \mathcal{N}_-}} c_n \left[\left(1 - i \frac{\omega_n - n}{\tilde{m}} \right) \psi_n^I - i \left(1 + i \frac{\omega_n - n}{\tilde{m}} \right) \psi_n^{I+4} \right] (a_{-n}^I + ia_{-n}^{I+4}) \quad (109)
\end{aligned}$$

with the corresponding complex conjugated expression for Q^\dagger .

5.4.3 The superalgebra

Adding up the appropriate contributions from (104) and (109) corresponding to the particular (n, n) -brane under consideration, one obtains the supercharges representing the conserved $\mathcal{N} = 2$ supersymmetry structure of the open string theory.

The anticommutators are found to be

$$\{Q, Q^\dagger\} = (8X^+) H^{\text{open}} \quad (110)$$

$$\{Q, Q\} = 8\pi\tilde{m} \sum_{i \in \mathcal{D}_-} \left((z_0^i)^2 - (z_\pi^i)^2 \right) \quad (111)$$

which completes our discussion of the open string superalgebra.

6 Spacetime supersymmetry and boundary states

In this section we will study the branes introduced in sections 4 and 5 from a closed string perspective by formulating them in terms of boundary states. This will on the one hand confirm our previous results, but is on the other hand in particular also suitable for a discussion of preserved spacetime supersymmetries. As a main result, the spacetime filling $(4, 4)$ -brane will be seen to be maximally spacetime supersymmetric. This can be understood in direct analogy to the other limiting case of the $(0, 0)$ -instanton. To have a more straightforward comparison with the constructions known for example from [23], we will use a formulation based on Green-Schwarz spinors in the closed string channel.

6.1 Gluing conditions

By using the standard procedure as for example explained in [1] or in the context of branes in the plane wave background in [30, 23] one translates the open string boundary to the corresponding closed string gluing conditions. For the bosonic fields we obtain from (55) and (71)

$$0 = (x^r(\tau, \sigma) - y_0^r)|_{\tau=0} ||\mathcal{B}\rangle \quad (112)$$

$$0 = \left(\partial_\tau x^I(\tau, \sigma) + i \left(b^{\hat{I}} x^{\hat{I}}(\tau, \sigma) + k^I \right) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \quad (113)$$

with $r \in \mathcal{D}$ and $I \in \mathcal{N}$. For the fermions, on the other hand, we have along the Dirichlet directions with $r \in \mathcal{D}$

$$0 = \left(\psi_+^r(\tau, \sigma) - i\rho\psi_-^r(\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle \quad (114)$$

and for the Neumann directions

$$0 = \partial_\sigma \left(\psi_+^I(\tau, \sigma) + i\rho\psi_-^I(\tau, \sigma) \right) + i(b^{\hat{I}} - \rho m) \left(\psi_+^{\hat{I}}(\tau, \sigma) - i\rho\psi_-^{\hat{I}}(\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle \quad (115)$$

with $I \in \mathcal{N}_-$. For $I \in \mathcal{N}_+$ one has to interchange $m \leftrightarrow -m$ and the parameter $\rho = \pm 1$ distinguishes as before between the brane / antibrane cases.

Translating these conditions to relations between Green-Schwarz fermionic fields by applying the results mentioned in section 2, one derives the gluing conditions

$$0 = \eta^* \Gamma^j \left(\tilde{S}(\tau, \sigma) - i\rho S(\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle; \quad 0 = \eta \Gamma^{\bar{j}} \left(\tilde{S}(\tau, \sigma) - i\rho S(\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle \quad (116)$$

along the Dirichlet directions with $j, \bar{j} \in \mathcal{D}_-$ and

$$0 = \eta^* \Gamma^{\hat{j}} \left(\partial_\sigma \left(\tilde{S} + i\rho S \right) (\tau, \sigma) + i \left(b^{\hat{j}} - m\rho\Pi \right) \left(\tilde{S} - i\rho S \right) (\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle \quad (117)$$

$$0 = \eta \Gamma^{\hat{\bar{j}}} \left(\partial_\sigma \left(\tilde{S} + i\rho S \right) (\tau, \sigma) + i \left(b^{\hat{\bar{j}}} - m\rho\Pi \right) \left(\tilde{S} - i\rho S \right) (\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle \quad (118)$$

along the Neumann directions with $j, \bar{j} \in \mathcal{N}_-$ and $b^j = b^{\bar{j}}$ as before.

To combine the fermionic gluing conditions to a single formula we define matrices \mathcal{R}, \mathcal{T} by the following requirements

$$\eta^* \Gamma^i \mathcal{R} = \eta^* \Gamma^i; \quad \eta \Gamma^{\bar{i}} \mathcal{R} = \eta \Gamma^{\bar{i}} \quad (119)$$

$$\eta^* \Gamma^i \mathcal{T} = b^i \eta^* \Gamma^i; \quad \eta \Gamma^{\bar{i}} \mathcal{T} = b^i \eta \Gamma^{\bar{i}} \quad (120)$$

along the Neumann directions with $i, \bar{i} \in \mathcal{N}_-$ and

$$\eta^* \Gamma^r \mathcal{R} = \eta^* \Gamma^r \mathcal{T} = 0; \quad \eta \Gamma^{\bar{r}} \mathcal{R} = \eta \Gamma^{\bar{r}} \mathcal{T} = 0 \quad (121)$$

for the Dirichlet directions with $r, \bar{r} \in \mathcal{D}_-$. These matrices especially fulfil

$$\mathcal{R}^2 = \mathcal{R}; \quad [\mathcal{R}, \mathcal{T}] = [\mathcal{R}, \Pi] = [\mathcal{T}, \Pi] = 0. \quad (122)$$

By using \mathcal{R} and \mathcal{T} the fermionic gluing conditions simplify to the single expression

$$0 = \left(\mathcal{R} \partial_\sigma \left(\tilde{S} + i\rho S \right) (\tau, \sigma) + i (\mathcal{T} - m\rho\Pi) \left(\tilde{S} - i\rho S \right) (\tau, \sigma) \right) \Big|_{\tau=0} ||\mathcal{B}\rangle \rangle. \quad (123)$$

6.1.1 The boundary state of the (n, n) -brane

By using the closed string mode expansions derived in [18], the previously established field-gluing conditions translate into relations between closed string modes acting on the boundary states. We use the conventions of [23], summarised in their appendix A. The bosonic conditions become

$$0 = (x_0^i - y_0^i) ||\mathcal{B}\rangle\rangle; \quad 0 = (\alpha_n^i - \tilde{\alpha}_{-n}^i) ||\mathcal{B}\rangle\rangle \quad (124)$$

for $i \in \mathcal{D}$ and

$$0 = \left(P_0^I + i \left(b^{\hat{I}} x_0^{\hat{I}} + k^I \right) \right) ||\mathcal{B}\rangle\rangle; \quad 0 = \left(\alpha_n^I + \frac{\omega_n + b^{\hat{I}}}{\omega_n - b^{\hat{I}}} \tilde{\alpha}_{-n}^{\hat{I}} \right) ||\mathcal{B}\rangle\rangle \quad (125)$$

with $I \in \mathcal{N}$. The fermionic gluing conditions translate into

$$0 = \left(\tilde{S}_0 - i\rho S_0 \right) ||\mathcal{B}\rangle\rangle \quad (126)$$

$$0 = \left(\tilde{S}_n - i\rho \frac{\omega_n - \rho m \Pi}{n} \left(1 - \frac{2\omega_n}{\omega_n - \mathcal{T}} \mathcal{R} \right) S_{-n} \right) ||\mathcal{B}\rangle\rangle. \quad (127)$$

Finally, by using the following zero mode combinations from [18, 23]

$$a^r = \frac{1}{\sqrt{2m}} (p_0^r + imx_0^r); \quad \bar{a}^r = \frac{1}{\sqrt{2m}} (p_0^r - imx_0^r) \quad (128)$$

the bosonic zero mode gluing conditions furthermore take on the structure

$$0 = \left(\bar{a}^i - a^i + i\sqrt{2m}y^i \right) ||\mathcal{B}\rangle\rangle \quad (129)$$

$$0 = \left(\bar{a}^I + \frac{m + b^{\hat{I}}}{m - b^{\hat{I}}} a^{\hat{I}} + i \frac{\sqrt{2m}k^{\hat{I}}}{m - b^{\hat{I}}} \right) ||\mathcal{B}\rangle\rangle. \quad (130)$$

After determining the closed string gluing conditions in (124)-(130) it is now straightforward to write down the corresponding boundary state up to an overall normalisation. This normalisation $\mathcal{N}_{(n,n)}$ is obtained from the results presented in section 7 in the standard procedure by comparing a suitable closed string boundary state overlap with the corresponding open string one loop partition function. As in the instanton case from [23], the normalisation $\mathcal{N}_{(n,n)}$ turns out to be

$$\mathcal{N}_{(n,n)} = (4\pi m)^2 \quad (131)$$

up to an irrelevant overall constant phase. With the gluing conditions (124)-(130), the boundary state takes on the form

$$\begin{aligned} ||\mathcal{B}\rangle\rangle = & \mathcal{N}_{(n,n)} \exp \left[\sum_{r=1}^{\infty} \sum_{i \in \mathcal{D}} \frac{1}{\omega_r} \alpha_{-r}^i \tilde{\alpha}_{-r}^i + \sum_{r=1}^{\infty} \sum_{I \in \mathcal{N}} \frac{1}{\omega_r} \frac{\omega_r + b^I}{\omega_r - b^I} \alpha_{-r}^I \tilde{\alpha}_{-r}^I \right. \\ & \left. + i\rho \sum_{r=1}^{\infty} \sum_{a,b} \left(\frac{\omega_r - m\rho\Pi}{r} \left(1 - \frac{2\omega_r}{\omega_r - \mathcal{T}} \mathcal{R} \right) \right)_{ab} \tilde{S}_{-r}^a S_{-r}^b \right] ||\mathcal{B}_0\rangle\rangle \end{aligned} \quad (132)$$

with

$$||\mathcal{B}_0\rangle\rangle = \prod_{I \in \mathcal{N}} \prod_{i \in \mathcal{D}} (\mathcal{B}_0^I \mathcal{B}_0^i) |0, \rho\rangle_f \quad (133)$$

and

$$\mathcal{B}_0^i = \exp \left[\left(\frac{1}{2} a^{\hat{i}} a^{\hat{i}} - i\sqrt{2m} y^{\hat{i}} a^{\hat{i}} \right) \right] e^{-\frac{m}{2} y^{\hat{i}} y^{\hat{i}}} \quad (134)$$

$$\mathcal{B}_0^I = \exp \left[- \left(\frac{1}{2} \frac{m + b^{\hat{I}}}{m - b^{\hat{I}}} a^{\hat{I}} a^{\hat{I}} + i \frac{\sqrt{2m} k^{\hat{I}}}{m - b^{\hat{I}}} a^{\hat{I}} \right) \right] e^{-\frac{m}{2} \frac{k^{\hat{I}} k^{\hat{I}}}{(b^{\hat{I}})^2 - m^2}}. \quad (135)$$

The fermionic vacuum state $|0, \rho\rangle_f$ is finally determined by the condition (126), compare for example with [23].

6.2 Spacetime supersymmetry

In this section we will determine the preserved (spacetime) supersymmetries of the boundary state (132). Our discussion from the open string point of view in section 5 together with the considerations from section 2 ensures at least two preserved supersymmetries on (132). Under certain conditions, however, some (n, n) -branes preserve additional supercharges. A certain class of $(4, 4)$ -branes, for example, will be seen to be even maximally supersymmetric.

In the conventions of [23] the (dynamical) supersymmetries of the plane wave background take on the form

$$\sqrt{2P^+} Q = \sum_r \left[p_0^r \gamma^r S_0 - m x_0^r \gamma^r \Pi \tilde{S} + \sum_{n \neq 0} c_n \left(\gamma^r \alpha_{-n}^r S_n + i \frac{\omega_n - n}{m} \gamma^r \Pi \tilde{\alpha}_{-n}^r \tilde{S}_n \right) \right] \quad (136)$$

$$\sqrt{2P^+} \tilde{Q} = \sum_r \left[p_0^r \gamma^r \tilde{S}_0 + m x_0^r \gamma^r \Pi S + \sum_{n \neq 0} c_n \left(\gamma^r \tilde{\alpha}_{-n}^r \tilde{S}_n - i \frac{\omega_n - n}{m} \gamma^r \Pi \alpha_{-n}^r S_n \right) \right] \quad (137)$$

and the conservation of supersymmetries by the boundary state (132) is expressed by

$$0 = P \left(Q + i\rho M \tilde{Q} \right) ||\mathcal{B}\rangle\rangle \quad (138)$$

with a constant $SO(8)$ -spinor matrix M and a suitable projector P whose (maximal) rank equals the number of preserved (dynamical) supersymmetries.

By using (124) - (127) we derive conditions for the matrices M and P as follows. From the zero modes along either the Dirichlet or Neumann directions from (124) and (125) we obtain

$$0 = P(1 - M) \quad \Leftrightarrow \quad PM = P. \quad (139)$$

From the nonzero modes along the Dirichlet directions from (124) we have

$$0 = P\gamma^i \left(\left(1 + \rho \frac{\omega_n - n}{m} \Pi \right) S_n + i\rho \left(1 - \rho \frac{\omega_n - n}{m} \Pi \right) \tilde{S}_{-n} \right) ||\mathcal{B}\rangle\rangle \quad (140)$$

and with (127)

$$0 = P\gamma^i \mathcal{R}. \quad (141)$$

From the Neumann directions with the gluing conditions (125) one furthermore derives

$$0 = P\gamma^{\hat{I}} \left(\frac{\omega_n + b^{\hat{I}}}{\omega_n - b^{\hat{I}}} \left(1 - \rho \frac{\omega_n - n}{m} \Pi \right) S_{-n} - i\rho \left(1 + \rho \frac{\omega_n - n}{m} \Pi \right) \tilde{S}_n \right) ||\mathcal{B}\rangle\rangle \quad (142)$$

from which

$$0 = P\gamma^{\hat{I}} \left[\omega_n (1 - \mathcal{R}) + \mathcal{R} (b^{\hat{I}} - \mathcal{T}) \right] \quad (143)$$

results by using (127). As (143) is required to hold for all n the conditions for preserved supersymmetries finally become

$$\begin{aligned} 0 &= P\gamma^I (1 - \mathcal{R}); & 0 &= P\gamma^{\hat{I}} (b^{\hat{I}} - \mathcal{T}) \\ 0 &= P\gamma^i \mathcal{R} \end{aligned} \quad (144)$$

with $I \in \mathcal{N}$ and $i \in \mathcal{D}$.

From (144) we can read off the number of conserved supersymmetries for (n, n) -branes

with the present boundary conditions. To start with, for $n = 0$ one obtains the $(0, 0)$ -instanton from [22, 23]. It has only Dirichlet directions and from (121) we furthermore have $\mathcal{R} = 0$, that is, P is of maximal rank, implying a maximally supersymmetric brane. This is of course exactly the result of [22, 23].

The remaining branes preserve at least the $\mathcal{N} = 2$ supersymmetry structure discussed in section 5 from an open string point of view. Here this subalgebra is obtained by the projector

$$P = |\eta\rangle\langle\eta^*| + |\eta^*\rangle\langle\eta|, \quad (145)$$

using the constant spinor η defined in section 2. For the $(1, 1)$ and $(3, 3)$ -branes and in case of pairwise different b^i for the $(2, 2)$ and $(4, 4)$ -branes these exhaust the conserved supersymmetries.

For homogenous boundary conditions along the Neumann directions, that is, by using the same parameter b for all Neumann blocks, there, however, appear additional supersymmetries for the $(2, 2)$ and the $(4, 4)$ brane beyond the $\mathcal{N} = 2$ subalgebra. Using the matrices \mathcal{R}, \mathcal{T} the situation of homogenous boundary conditions translates into

$$\mathcal{T} = b\mathcal{R}, \quad (146)$$

simplifying (144) accordingly. Evaluating these conditions with (146), the $(2, 2)$ brane is found to be quarter supersymmetric, that is, it preserves 4 supersymmetries and the $(4, 4)$ brane with $\mathcal{R} = 1$ and no Dirichlet directions along the transverse coordinates becomes finally even maximally supersymmetric.

Using the classification of [23], see also [22], the (n, n) -branes all belong to the class II branes. Our $(4, 4)$ therefore adds a maximally supersymmetric brane to this family, containing so far only the other extremal case of the $(0, 0)$ instanton and the $(4, 0)$, $(0, 4)$ branes as half supersymmetric branes.

6.3 Boundary conditions with longitudinal flux \mathcal{F}_{I+}

In this section we will briefly discuss how to realise deformed Neumann boundary conditions as in (71) by switching on a nonzero flux \mathcal{F}_{I+} . In the context of plane wave physics this has been first discussed in [29] and later on applied in particular in [22, 23, 34].

In the presence of a boundary condensate Neumann conditions read

$$\partial_\sigma X^r = \mathcal{F}_s^r \partial_\tau X^s \quad (147)$$

at $\sigma = 0, \pi$. By switching on only particular longitudinal components of \mathcal{F} one obtains

$$\partial_\sigma x^r = \mathcal{F}_+^r \partial_\tau X^+ \sim \mathcal{F}_+^r P^+ \quad (148)$$

by using the standard lightcone gauge condition on X^+ . Choosing the flux \mathcal{F}_{I+} as a general affine function in X^I with appropriate constant factors one obtains from (148) the boundary conditions (71), compare again with [29].

For boundary fields F, G fulfilling the requirements for $\mathcal{N} = 2$ supersymmetry and integrability the boundary conditions (20) and (21) were seen in section 4 to be independent of the fermionic fields and take on as shown above the standard form for Neumann boundary conditions in the presence of a particular boundary condensate. Nevertheless, the fermionic boundary conditions (22), (23) respectively (72) and (73) differ clearly from the conditions usually employed for the fermionic fields in the presence boundaries. It would be very interesting to obtain a deeper understanding of these conditions and their relation to the flux \mathcal{F}_{I+} from (148) for example by considerations along the lines of [10].

Before discussing the open/closed duality in section 7, we use (148) to explain the relation between the open string quantities \tilde{b}, \tilde{m} and \tilde{k} and their closed string relatives b, m, k . As discussed in [30, 23, 32] to which we refer for a detailed treatment, one needs to apply different lightcone gauge conditions in the open respectively closed string sectors to deal with branes of the same structure in both cases. In (148) this effectively amounts to interchange the roles of P^+ and the lightcone separation X^+ of the branes under consideration. As discussed in [30] it follows immediately from this observation that m, b, k are related to the corresponding open string quantities by

$$\tilde{m} = mt; \quad \tilde{b} = bt; \quad \tilde{k} = kt \quad (149)$$

with

$$t = \frac{X^+}{2\pi P^+}. \quad (150)$$

The number t is the modular parameter to appear in section 7 where also the relations (149) will be put to use.

6.4 The $b \rightarrow m$ limit.

To discuss the limiting situation of $b = m$ excluded in the previous discussion we briefly reconsider the local boundary field $\Sigma_\sigma(\tau)$ introduced in section 4. This will especially

also establish the maximal supersymmetry of the $(4, 4)$ brane in the open string sector which so far has been done only for the particular $\mathcal{N} = 2$ subalgebra discussed in section 2.

From the supercurrents derived in [18], used here in the conventions of [23], the condition for conserved spacetime supercharges in the open sector corresponding to (138) is given by

$$\partial_\tau \Sigma_\sigma = P \left[\left(\partial_- x^a \gamma^a S + \tilde{m} x^a \gamma^a \Pi \tilde{S} \right) + M \left(-\partial_+ x^a \gamma^a \tilde{S} + \tilde{m} x^a \gamma^a \Pi S \right) \right], \quad (151)$$

following as before [1]. The equation (151) is again understood to be evaluated at the boundaries $\sigma = 0, \pi$ and for the case of the $(4, 4)$ brane to which we restrict attention here one furthermore has $P = M = 1$.

By using the bosonic boundary conditions (71) and

$$0 = \left(\partial_\tau \left(\tilde{S} - S \right) - \left(\tilde{b} - \tilde{m} \Pi \right) \left(\tilde{S} + S \right) \right) \Big|_{\sigma=0, \pi} \quad (152)$$

corresponding to (123), we derive the following local boundary field

$$\Sigma_\pi(\tau) = \sum_I \left[\left(X^I + \frac{\tilde{k}^I}{\tilde{b} - \tilde{m} \Pi} \right) \gamma^I \left(S - \tilde{S} \right) \right]_{\sigma=\pi}. \quad (153)$$

As it fulfills (151), the open string theory for the $(4, 4)$ brane preserves the maximal supersymmetry as expected from the boundary state treatment.

From (153) it is furthermore apparent that the $(4, 4)$ remains maximally supersymmetric in the $\tilde{b} \rightarrow \tilde{m}$ limit in case of $\tilde{k} = 0$ corresponding to the choice $C^i = 0$ in (69).

It is worth pointing out that the bosonic boundary conditions (71) take on in this limit the structure used in [34] in an alternative construction of (n, n) -branes. There the authors show from an open string point of view that the common fermionic boundary conditions

$$0 = \left(\tilde{S} - MS \right) \Big|_{\sigma=0, \pi} \quad (154)$$

with a matrix M as defined in section 3 together with the bosonic boundary conditions

$$\partial_\sigma X^I = \pm m X^I; \quad \partial_\sigma X^{I+4} = \mp m X^{I+4} \quad (155)$$

with $I \in \mathcal{N}_-$ lead to (n, n) -branes ($n = 1, \dots, 4$) which preserve 4 spacetime supersymmetries. This is expressed by the projectors

$$P = \frac{1 \pm M \Pi}{2} \quad (156)$$

in the conditions (151) and (138).

7 Open-Closed duality

In this section we consider an important consistency check for the (n, n) -boundary states constructed in section 6 by testing the equality of the closed string boundary state overlap

$$\mathcal{A}(t) = \langle \langle \mathbf{b}, \mathbf{k}, \mathbf{y}_2 | | e^{-2\pi t H^{\text{closed}} P^+} | | \mathbf{b}, \mathbf{k}, \mathbf{y}_1 \rangle \rangle \quad (157)$$

and the one loop open string partition function

$$Z(\tilde{t}) = \text{Tr} \left[e^{-\frac{X^+}{2\pi} H^{\text{open}} \tilde{t}} \right]. \quad (158)$$

The trace in (158) runs over the states of an open string spanning between branes with boundary conditions corresponding to the boundary states in (157). In the context of plane wave physics this consistency check was first considered in [30, 23] to which we refer for a detailed discussion. Here we only note that the modular parameters are related by

$$\tilde{t} = \frac{1}{t} \quad (159)$$

and the field parameters b^i, k^i, m translate as discussed in section 6.3.

We will express (157) and (158) in terms of special functions defined in [30, 23] as m -dependent deformations of the f -functions defined in [41] by Polchinski and Cai.

For open strings spanning between two (n, n) -branes of the same type there are fermionic zero modes commuting with the corresponding open string Hamiltonian. As explained for example in [30] these modes lead to vanishing open string partition functions. In the closed string sector this result is confirmed by considering the zero mode part overlap which is also found to vanish, see again [31, 30, 23].

To obtain a nontrivial behaviour we consider the situation of a brane-antibrane configuration. From (158) we have for the open string partition function along each complex pair of Dirichlet directions

$$Z_{x^i, x^{i+4}}(\tilde{t}) = e^{-\frac{\tilde{t}\tilde{m}}{2 \sinh(\tilde{m}\pi)}} \sum_{j=i, i+4} (\cosh(\tilde{m}\pi) (y_2^j y_2^j + y_1^j y_1^j) - 2y_2^j y_1^j) \frac{\hat{g}_4^{(\tilde{m})}(\tilde{q})}{\left(f_1^{(\tilde{m})}(\tilde{q})\right)^2} \quad (160)$$

with $\tilde{q} = e^{-2\pi\tilde{t}}$. For a pair of Neumann directions we deduce analogously

$$Z_{x^I, x^{I+4}}(\tilde{t}) = e^{-\tilde{m}\tilde{t}} \sum_{J=I, I+4} \frac{\tanh \frac{\tilde{m}\pi}{2}}{(\tilde{b}^J)^2 - \tilde{m}^2} \tilde{k}^J \tilde{k}^J \frac{\hat{g}_4^{(\tilde{m})}(\tilde{q})}{\left(f_1^{(\tilde{m})}(\tilde{q})\right)^2}. \quad (161)$$

For the boundary state overlap (157) one derives

$$\mathcal{A}_{x^i, x^{i+4}}(t) = \exp \left[- \sum_{j=i, i+4} \left(\frac{m(1+q^m)(y_1^j y_1^j + y_2^j y_2^j)}{2(1-q^m)} + \frac{2mq^{\frac{m}{2}} y_1^j y_2^j}{1-q^m} \right) \right] \frac{g_2^{(m)}(q)}{\left(f_1^{(m)}(q)\right)^2} \quad (162)$$

along each pair of Dirichlet directions and

$$\mathcal{A}_{x^I, x^{I+4}}(t) = \exp \left[- \sum_{J=I, I+4} \frac{mk^J k^J}{(b^J)^2 - m^2} \frac{1 - q^{\frac{m}{2}}}{1 + q^{\frac{m}{2}}} \right] \frac{g_2^{(m)}(q)}{\left(f_1^{(m)}(q)\right)^2} \quad (163)$$

along a pair of Neumann directions by using in both cases the normalisation (131). The zero mode prefactors in (162), (163) are for example calculated by inserting a complete set of coherent states as explained in [31].

From the modular transformations properties

$$f_1^{(m)}(q) = f_1^{(\tilde{m})}(\tilde{q}); \quad g_2^{(m)}(q) = \hat{g}_4^{(\tilde{m})}(\tilde{q}) \quad (164)$$

derived in [30, 23], the open string partition functions (160) and (161) are seen to be equal to the corresponding closed string boundary state overlaps (162) and (163). By this, the (n, n) -branes pass this important consistency check.

8 Conclusions

Starting with a boundary Lagrangian containing fermionic boundary excitations defined in analogy to the settings in [1, 2, 4] and [6, 7] from the context of integrable boundary field theories and matrix factorisations in string theory, we have constructed new integrable and supersymmetric branes in the plane wave background of type (n, n) . As a main result, the limiting case of the spacetime filling $(4, 4)$ -brane was shown to be maximally supersymmetric. This is in analogy to the other extremal case of the $(0, 0)$ -instanton from [22, 23].

The new branes were constructed in the open and closed string picture, leading to consistent results in both sectors. The branes pass in particular the open/close-duality check of the equality of open-string one loop partition functions and corresponding boundary state overlaps, compare with [30, 23].

Whereas the deformed bosonic boundary conditions along the Neumann directions can be understood as a coupling to a nonzero flux \mathcal{F}_{+I} , a statement also supported by the

correct reproduction of the relation between gauge-dependent field parameters in the open and closed sector as implied by duality, the situation for the fermionic sector is less clear. It was demonstrated that for integrable branes the boundary fermions are consistently determined by the bulk fields restricted to the boundaries. However, a more geometric understanding of the resulting deformed boundary conditions in the fermionic sector, for example along the lines of [9, 10], remains desirable.

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A LG vs GS spinors

In this Appendix we supply some additional details about the identifications between Landau-Ginzburg and Green-Schwarz fermions as briefly discussed in section 2. The identifications (8), (9) or the inverted expressions

$$\psi_-^i = \frac{1}{2}\eta^{*a}\Gamma_{ab}^i S^b \quad \quad \bar{\psi}_-^{\bar{i}} = \frac{1}{2}\eta^a\Gamma_{ab}^{\bar{i}} S^b \quad (165)$$

$$\psi_+^i = \frac{1}{2}\eta^{*a}\Gamma_{ab}^i \tilde{S}^b \quad \quad \bar{\psi}_+^{\bar{i}} = \frac{1}{2}\eta^a\Gamma_{ab}^{\bar{i}} \tilde{S}^b \quad (166)$$

can be geometrically interpreted as follows [11]. The choice of a complex structure in the definition of the Landau-Ginzburg Lagrangian (4) figures out a $SU(4)$ subgroup of the $SO(8)$ in whose spinor representations the standard Green-Schwarz spinors reside. Under this subgroup these representations decompose into

$$\mathbf{8}_- \rightarrow \mathbf{4} + \bar{\mathbf{4}} \quad (167)$$

and the summands correspond to the spinor fields in (4) carrying a vector index. As the superpotential (5) already breaks the $SO(8)$ -background symmetry present in flat space down to $SO(4) \times SO(4) \times \mathbb{Z}_2$, the complex structure used in the previous argument actually picks out the diagonal $SO(4)$ subgroup of this product. For this reduced symmetry group the fields $\psi_\pm^i, \bar{\psi}_\pm^{\bar{i}}$ transform in the same representation, explaining the seemingly strange index structure of the equations of motion (7).

Before discussing the $\mathcal{N} = (2, 2)$ worldsheet supersymmetry, we briefly establish the existence of the spinor η with the requirements (10). Using the properties of the complex Dirac matrices $\Gamma^i, \Gamma^{\bar{i}}$, the spinor η is immediately determined to

$$\eta = \Gamma_{\alpha} \Gamma_{\bar{2}} \Gamma_{\bar{3}} \Gamma_{\bar{4}} (\mathbb{1} - \Pi) \zeta \quad (168)$$

with a constant real spinor $\zeta = \zeta^*$ of appropriate norm. For example by employing the explicit spinor representation presented in chapter 5 of [31] one can show that the matrix $\Gamma_{\bar{1}} \Gamma_{\bar{2}} \Gamma_{\bar{3}} \Gamma_{\bar{4}} (\mathbb{1} - \Pi)$ is of *real* rank one, that is, η is actually unique up to a sign. Finally, by using

$$\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \prod_{i=1}^4 \frac{\Gamma^i + \Gamma^{\bar{i}}}{\sqrt{2}} \quad (169)$$

the condition $\eta^* = -\Pi \eta$ becomes

$$\eta^* = -\frac{1}{4} \Gamma^{\bar{1}} \Gamma^{\bar{2}} \Gamma^{\bar{3}} \Gamma^{\bar{4}} \eta = -\frac{1}{4} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \eta. \quad (170)$$

The lowest weight $su(4)$ state η is therefore essentially related to the corresponding highest weight state by complex conjugation.

A.1 $\mathcal{N} = (2, 2)$ supersymmetry

In the following we will derive the relations (13), (14) between the $\mathcal{N} = (2, 2)$ worldsheet supersymmetry and the spacetime supercharges from the Green-Schwarz formulation. This will in particular also lead to an explicit confirmation of the related group theoretical discussion in [23].

The supercurrents for the plane-wave Landau-Ginzburg model described by (4) with superpotential (5) are given by [26]

$$G_{\pm}^0 = g_{i\bar{i}} \partial_{\pm} \bar{z}^{\bar{i}} \psi_{\pm}^i \pm m \bar{\psi}_{\mp}^{\bar{i}} \bar{z}^{\bar{i}} \quad G_{\pm}^1 = \mp g_{i\bar{i}} \partial_{\pm} \bar{z}^{\bar{i}} \psi_{\pm}^i + m \bar{\psi}_{\mp}^{\bar{i}} \bar{z}^{\bar{i}} \quad (171)$$

$$\bar{G}_{\pm}^0 = g_{i\bar{i}} \partial_{\pm} z^i \bar{\psi}_{\pm}^{\bar{i}} \pm m z^i \psi_{\mp}^i \quad \bar{G}_{\pm}^1 = \mp g_{i\bar{i}} \partial_{\pm} z^i \bar{\psi}_{\pm}^{\bar{i}} + m z^i \psi_{\mp}^i \quad (172)$$

and lead to the conserved charges

$$Q_{\pm} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left(g_{i\bar{j}} \partial_{\pm} \bar{z}^{\bar{j}} \psi_{\pm}^i \pm m \bar{\psi}_{\mp}^{\bar{j}} \bar{z}^{\bar{j}} \right) \quad (173)$$

$$\bar{Q}_{\pm} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left(g_{i\bar{j}} \bar{\psi}_{\pm}^{\bar{j}} \partial_{\pm} z^i \pm m \psi_{\mp}^i z^i \right) \quad (174)$$

representing the $\mathcal{N} = (2, 2)$ worldsheet supersymmetry. Using the identifications (8), (9) and

$$\gamma^r \eta = -i\gamma^{r+4} \eta; \quad \gamma^r \eta^* = i\gamma^{r+4} \eta^* \quad (175)$$

from (10), we for example deduce

$$Q_+ = \frac{1}{4\pi} \int_0^{2\pi} d\sigma \left(g_{i\bar{j}} \partial_+ \bar{z}^{\bar{j}} \left(\eta^* \Gamma^i \tilde{S} \right) + m \bar{z}^{\bar{j}} \left(\eta \Gamma^{\bar{j}} S \right) \right) \quad (176)$$

$$= \frac{\eta^*}{2\pi} \int_0^{2\pi} d\sigma \left(\partial_+ x^I \gamma^I \tilde{S} + m x^I \gamma^I \Pi S \right). \quad (177)$$

Comparing this with the expressions for the dynamical spacetime supercharges derived in [18], used here in the conventions of [23], we deduce

$$\frac{Q_+}{\sqrt{2p^+}} = \eta^* \tilde{Q} = (\eta^*)_{\dot{\alpha}} \tilde{Q}_{\dot{\alpha}}, \quad (178)$$

implicitly using the negative $SO(8)$ chiralities of the spinors S, \tilde{S} . In a similar way one expresses the remaining supersymmetries as linear combinations of the spacetime charges as given in (13), (14).

To relate this result to the discussion of section 5 in [23], we only have to note that the condition (10) requires η to be the bottom state discussed in [23], whereas η^* is the corresponding top-state as established beforehand at the end of the last section.

B Integrability

In this appendix we present some additional information about the integrable structure underlying the plane wave theory. For the massive Ising model the higher spin currents responsible for integrability were written down in [40] and are given by

$$\begin{aligned} T_{n+1}^f &= g_{i\bar{i}} \bar{\psi}_+^{\bar{i}} \partial_+^n \psi_+^i; & \theta_{n-1}^b &= -g_{i\bar{i}} \psi_-^i \partial_+^n \bar{\psi}_-^{\bar{i}} \\ \bar{T}_{n+1}^f &= g_{i\bar{i}} \bar{\psi}_-^{\bar{i}} \partial_-^n \psi_-^i; & \bar{\theta}_{n-1}^b &= -g_{i\bar{i}} \psi_+^i \partial_-^n \bar{\psi}_+^{\bar{i}} \end{aligned} \quad (179)$$

by concentrating on the for a theory defined on $S^1 \times \mathbb{R}$ relevant cases. The corresponding bosonic currents are found to equal

$$\begin{aligned} T_{2n}^b &= g_{i\bar{i}} \partial_+^n \bar{z}^{\bar{i}} \partial_+^n z^i; & \theta_{2n-2}^b &= -m^2 g_{i\bar{i}} \partial_+^{n-1} \bar{z}^{\bar{i}} \partial_+^{n-1} z^i \\ \bar{T}_{2n}^b &= g_{i\bar{i}} \partial_-^n \bar{z}^{\bar{i}} \partial_-^n z^i; & \bar{\theta}_{2n-2}^b &= -m^2 g_{i\bar{i}} \partial_-^{n-1} \bar{z}^{\bar{i}} \partial_-^{n-1} z^i \end{aligned} \quad (180)$$

and the integrable currents for the plane wave theory are given by a suitable combination of (179) and (180). Appearing relative prefactors might for example be determined by requiring the cancellation of separate normal ordering constants in (179) and (180) in the quantum theory.

Treating a free theory, there are nevertheless many different fluxes like (180). They can for example be obtained by taking the parts along single real directions in (180) and recombining them in various ways. This leads to additional conserved higher spin bulk currents, but most choices are incompatible with the complex structure chosen in the Lagrangian (4).

Our decision to consider the special combinations (37)-(40) is especially based on the observation that these currents appear as limits of the highly nontrivial higher spin currents of the $\mathcal{N} = 2$ supersymmetric sine-Gordon model.

The first nontrivial higher spin currents for this theory were formulated in [42, 4]. In the language of a Landau-Ginzburg model with superpotential

$$W = -2ig \cos z + \text{const} \quad (181)$$

they can be found in [15]. Reintroducing the standard parameter ω and rescaling the coupling constant to $g \rightarrow -\frac{m}{\omega^2}$, the plane wave like theory with superpotential $W = imz^2$ is obtained from (181) in the $\omega \rightarrow 0$ limit.

Using the higher spin currents as presented in [15] we obtain furthermore for the first higher spin flux

$$\frac{T_4}{\omega^2} = 2 \left(\partial_+^2 \bar{z} \partial_+^2 z + i \partial_+ \bar{\psi}_+ \partial_+^2 \psi_+ \right) + o(\omega^2) \quad (182)$$

$$\frac{\theta_2}{\omega^2} = 2 \left(-m^2 \partial_+ \bar{z} \partial_+ z - im^2 \bar{\psi}_+ \partial_+ \psi_+ \right) + o(\omega^2). \quad (183)$$

The formulas presented in (37) and (38) and correspondingly in (39) and (40) differ from (182), (183) only in total derivative terms included to obtain manifestly real expressions.

In the boundary theory the currents (37)-(40) give rise to the conserved charge

$$I_3 = \int_0^\pi d\sigma \left(T_4 + \bar{T}_4 - \theta_2 - \bar{\theta}_2 \right) - \Sigma_\pi^{(3)}(t) + \Sigma_0^{(3)}(t) \quad (184)$$

with local boundary fields $\Sigma_{0,\pi}^{(3)}(t)$. The calculational strategy to determine these fields and the corresponding differential equations for F, G and the boundary potential B is

explained in detail in [15]. We omit the details here and present only the explicit form of the boundary current $\Sigma_\pi(t)$ along the Neumann directions. It is given by

$$\begin{aligned}
\Sigma_\pi^{(3)}(t) = & 4m^2 \partial_\tau \partial_i B z^i \bar{z}^\tau + 2m^2 \partial_\tau \partial_j B \bar{z}^\tau \bar{z}^j + 2m^2 \partial_i \partial_j B z^i z^j \\
& + 8\partial_i \partial_j B \partial_\tau z^i \partial_\tau \bar{z}^j + 4\partial_i \partial_j B \partial_\tau \bar{z}^\tau \partial_\tau \bar{z}^j + 4\partial_i \partial_j B \partial_\tau z^i \partial_\tau z^j \\
& + 6m^2 i \bar{\theta}_+^\tau \theta_-^i - 6im^2 \bar{\theta}_-^\tau \theta_+^i + 8i \partial_\tau \bar{\theta}_-^\tau \partial_\tau \theta_+^i - 8i \partial_\tau \bar{\theta}_+^\tau \partial_\tau \theta_-^i \\
& + 4mie^{i\beta} (\theta_+^i \partial_\tau \theta_+^i - \theta_-^i \partial_\tau \theta_-^i) - 4mie^{-i\beta} (\partial_\tau \bar{\theta}_+^\tau \bar{\theta}_+^\tau - \partial_\tau \bar{\theta}_-^\tau \bar{\theta}_-^\tau). \quad (185)
\end{aligned}$$

The conservation of a higher spin current like (184) leads to strong evidence for the integrability of the underlying field theory, but does clearly not constitute a proof. As mentioned in section 4.3 one might for the present model furthermore test the mode expansions and commutation relations of section 5 against the requirements derived in [1] for an integrable boundary theory. These are in particular the boundary Yang-Baxter equation, the unitarity requirement and the crossing symmetry which relates the open string mode identifications to the corresponding closed string gluing conditions by an analytic continuation in the so called rapidity variable. We will not spell out the details here, but mention that the modings derived in section 5 fulfil all the requirements presented in [1]. One might compare this also with the treatment of the massive Ising model in [1] and [24].

Finally, we want to comment on the number of boundary parameters in the Lagrangian (19) in case of integrable and supersymmetry preserving boundary conditions along a single Neumann direction. From (65) and (66) in section 5 we have three real parameters as obtained to first order in the bulk coupling constant for the $\mathcal{N} = 2$ sine-Gordon model in [4]. For the sine-Gordon model a calculation taking into account all order contributions reduces this number to a single boundary parameter as shown in [15]. In the case of present interest, however, contributions leading to these additional constraints vanish in the $\omega \rightarrow 0$ limit, compare especially with the quadratic form of (66) in comparison with the trigonometric boundary potential in [4, 15] and the discussion in section 4.3 of [15].

C Quantisation

In this appendix we supply some details of the quantisation process omitted beforehand in section 5. The required relations in the quantum theory are given by

$$[x^r(\tau, \sigma), p^s(\tau, \bar{\sigma})] = 4\pi i \delta^{rs} \delta(\sigma - \bar{\sigma}) \quad (186)$$

$$[x^r(\tau, \sigma), x^s(\tau, \bar{\sigma})] = 0 \quad (187)$$

$$[p^r(\tau, \sigma), p^s(\tau, \bar{\sigma})] = 0 \quad (188)$$

for the bosons and

$$\{\psi_+^a(\tau, \sigma), \psi_+^b(\tau, \bar{\sigma})\} = 2\pi \delta^{ab} \delta(\sigma - \bar{\sigma}) \quad (189)$$

$$\{\psi_-^a(\tau, \sigma), \psi_-^b(\tau, \bar{\sigma})\} = 2\pi \delta^{ab} \delta(\sigma - \bar{\sigma}) \quad (190)$$

$$\{\psi_+^a(\tau, \sigma), \psi_-^b(\tau, \bar{\sigma})\} = 0 \quad (191)$$

for the fermions, always understood to be evaluated for $0 < \sigma, \bar{\sigma} < \pi$. Choosing appropriate normalisations of the nonzero modes in the field expansions, the corresponding commutation relations take on the canonical form presented in section 5.1.1 and 5.2.1. The relations for the zero modes are deduced from that by using the contour integral method sketched in section 5.3.

For the fermions along Dirichlet directions with $a, b \in \mathcal{D}_-$ we have for example

$$\begin{aligned} \{\psi_+^a(\tau, \sigma), \psi_+^b(\tau, \bar{\sigma})\} &= \{\psi^a, \psi^b\} e^{-\tilde{m}(\sigma+\bar{\sigma})} + \sum_{r \neq 0} e^{ir(\sigma-\bar{\sigma})} + 2i \sum_{r \neq 0} c_r^2 \frac{r + i\tilde{m}}{\omega_r} \frac{\omega_r - r}{\tilde{m}} e^{ir(\sigma+\bar{\sigma})} \\ &= \{\psi^a, \psi^b\} e^{-\tilde{m}(\sigma+\bar{\sigma})} + \sum_{r \in \mathbb{Z}} e^{ir(\sigma-\bar{\sigma})} + \sum_{r \in \mathbb{Z}} \frac{i\tilde{m}}{r - i\tilde{m}} e^{ir(\sigma+\bar{\sigma})} \end{aligned} \quad (192)$$

with

$$\sum_{r \in \mathbb{Z}} \frac{i\tilde{m}}{r - i\tilde{m}} e^{ir(\sigma+\bar{\sigma})} = - \oint_{\mathcal{C}} dz \frac{e^{iz(\sigma+\bar{\sigma})}}{1 - e^{2\pi iz}} \frac{i\tilde{m}}{z - i\tilde{m}} = \frac{-2\pi \tilde{m} e^{-\tilde{m}(\sigma+\bar{\sigma})}}{1 - e^{-2\pi \tilde{m}}} \quad (193)$$

and

$$\sum_{r \in \mathbb{Z}} e^{ir(\sigma-\bar{\sigma})} = 2\pi \delta(\sigma - \bar{\sigma}), \quad 0 < \sigma, \bar{\sigma} < \pi. \quad (194)$$

By using the zero mode anticommutators (63) one obtains from that the required result (189).

For the bosons along a Neumann direction we have analogously

$$\begin{aligned} [x^I(\tau, \sigma), p^J(\tau, \bar{\sigma})] &= 2 \left(-[P^{\hat{I}}, Q^J] + [Q^{\hat{I}}, P^J] \right) \sqrt{(\tilde{b}^{\hat{I}})^2 - \tilde{m}^2} e^{\tilde{b}^{\hat{I}}(\sigma + \bar{\sigma})} \\ &\quad + 2i\delta^{IJ} \sum_{r \in \mathbb{Z} \setminus \{0\}} e^{ir(\sigma - \bar{\sigma})} + 2i\delta^{\hat{I}J} \sum_{r \in \mathbb{Z} \setminus \{0\}} e^{ir(\sigma + \bar{\sigma})} \frac{r - i\tilde{b}^{\hat{I}}}{r + i\tilde{b}^{\hat{I}}} \end{aligned} \quad (195)$$

with

$$\sum_{r \in \mathbb{Z}} e^{ir(\sigma + \bar{\sigma})} \frac{r - ib}{r + ib} = - \oint_{\mathcal{C}} dz \frac{e^{iz(\sigma + \bar{\sigma})}}{1 - e^{2\pi iz}} \frac{z - ib}{z + ib} = 4\pi b \frac{e^{b(\sigma + \bar{\sigma})}}{1 - e^{2\pi b}}. \quad (196)$$

From (186) we deduce (83).

Finally, for a fermionic field spanning along a Neumann direction with $I, J \in \mathcal{N}_-$ we obtain the equation

$$\begin{aligned} \{\psi_+^I(\tau, \sigma), \psi_+^J(\tau, \bar{\sigma})\} &= \{\psi^I, \psi^J\} e^{-\tilde{m}(\sigma + \bar{\sigma})} + 2 \left\{ \chi^{\hat{I}}, \tilde{\chi}^J \right\} \frac{\sqrt{(\tilde{b}^{\hat{I}})^2 - \tilde{m}^2} + \tilde{b}^{\hat{I}}}{\tilde{m}} e^{\tilde{b}^{\hat{I}}(\sigma + \bar{\sigma})} \\ &\quad + \delta^{IJ} \sum_{r \neq 0} e^{ir(\sigma - \bar{\sigma})} - i\tilde{m}\delta^{\hat{I}J} \sum_{r \neq 0} \frac{1}{r - i\tilde{m}} \frac{r - i\tilde{b}^{\hat{I}}}{r + i\tilde{b}^{\hat{I}}} e^{ir(\sigma + \bar{\sigma})} \end{aligned} \quad (197)$$

with in this case

$$-im \sum_{r \in \mathbb{Z}} \frac{1}{r - im} \frac{r - ib}{r + ib} e^{ir(\sigma + \bar{\sigma})} = \frac{2\pi m}{1 - e^{-2\pi m}} \frac{m - b}{m + b} e^{-m(\sigma + \bar{\sigma})} + \frac{4\pi m}{1 - e^{2\pi b}} \frac{b}{b + m} e^{b(\sigma + \bar{\sigma})}$$

confirming (85) and (86).

All other relations are either implied by the presented results or are established analogously.

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